2.3 Meaning of Gaussian Curvature

In the previous lecture we gave a formal definition for Gaussian curvature $K$ in terms of the differential of the gauss map, and also derived explicit formulas for $K$ in local coordinates. In this lecture we explore the geometric meaning of $K$.

2.3.1 A measure for local convexity

Let $M \subset \mathbb{R}^3$ be a regular embedded surface, $p \in M$, and $H_p$ be hyperplane passing through $p$ which is parallel to $T_p M$. We say that $M$ is \textit{locally convex} at $p$ if there exists an open neighborhood $V$ of $p$ in $M$ such that $V$ lies on one side of $H_p$. In this section we prove:

\textbf{Theorem 1.} If $K(p) > 0$ then $M$ is locally convex at $p$, and if $k(p) < 0$ then $M$ is not locally convex at $p$.

When $K(p) = 0$, we cannot in general draw a conclusion with regard to the local convexity of $M$ at $p$ as the following two exercises demonstrate:

\textbf{Exercise 2.} Show that there exists a surface $M$ and a point $p \in M$ such that $M$ is strictly locally convex at $p$; however, $K(p) = 0$ (\textit{Hint:} Let $M$ be the graph of the equation $z = (x^2 + y^2)^2$. Then may be covered by the Monge patch $X(u_1, u_2) := (u_1, u_2, ((u_1)^2 + (u_2))^2)$. Use the Monge Ampere equation derived in the previous lecture to compute the curvature at $X(0,0)$.)

\textbf{Exercise 3.} Let $M$ be the \textit{Monkey saddle}, i.e., the graph of the equation $z = y^3 - 3yx^2$, and $p := (0,0,0)$. Show that $K(p) = 0$, but $M$ is not locally convex at $p$. 

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1\textsuperscript{Last revised: October 31, 2004}
After a rigid motion we may assume that \( p = (0, 0, 0) \) and \( T_p M \) is the \( xy \)-plane. Then, using the inverse function theorem , it is easy to show that there exists a Monge Patch \((U, X)\) centered at \( p \), as the following exercise demonstrates:

**Exercise 4.** Define \( \pi : M \to \mathbb{R}^2 \) by \( \pi(q) := (q^1, q^2, 0) \). Show that \( d\pi_p \) is locally one-to-one. Then, by the inverse function theorem, it follows that \( \pi \) is a local diffeomorphism. So there exists a neighborhood \( U \) of \((0, 0)\) such that \( \pi^{-1} : U \to M \) is one-to-one and smooth. Let \( f(u_1, u_2) \) denote the coordinate of \( \pi^{-1}(u_1, u_2) \), and set \( X(u_1, u_2) := (u_1, u_2, f(u_1, u_2)) \). Show that \((U, X)\) is a proper regular patch.

The previous exercise shows that local convexity of \( M \) at \( p \) depends on whether or not \( f \) changes sign in a neighborhood of the origin. To examine this we need to recall the Taylor’s formula for functions of two variables:

\[
f(u_1, u_2) = f(0, 0) + \sum_{i=1}^{2} D_i f(0, 0) u_i + \frac{1}{2} \sum_{i,j=1}^{2} D_{ij}(\xi_1, \xi_2) u_i u_j,
\]

where \((\xi_1, \xi_2)\) is a point on the line connecting \((u_1, u_2)\) to \((0, 0)\).

**Exercise 5.** Prove the Taylor’s formula given above. (**Hints:** First recall Taylor’s formula for functions of one variable: \( g(t) = g(0) + g'(0) t + (1/2) g''(s) t^2 \), where \( s \in [0, t] \). Then define \( \gamma(t) := (tu_1, tu_2) \), set \( g(t) := f(\gamma(t)) \), and apply Taylor’s formula to \( g \). Then chain rule will yield the desired result.)

Next note that, by construction, \( f(0, 0) = 0 \). Further \( D_1 f(0, 0) = 0 = D_2 f(0, 0) \) as well. Thus

\[
f(u_1, u_2) = \frac{1}{2} \sum_{i,j=1}^{2} D_{ij}(\xi_1, \xi_2) u_i u_j.
\]

Hence to complete the proof of Theorem 1, it remains to show how the quantity on the right hand side of the above equation is influence by \( K(p) \). To this end, recall the Monge-Ampere equation for curvature:

\[
\det(\text{Hess } f(\xi_1, \xi_2)) = K(f(\xi_1, \xi_2))(1 + \|\text{grad } f(\xi_1, \xi_2)\|^2)^2.
\]
Now note that $K(f(0,0)) = K(p)$. Thus, by continuity, if $U$ is a sufficiently small neighborhood of $(0,0)$, the sign of $\det(\text{Hess } f)$ agrees with the sign of $K(p)$ throughout $U$.

Finally, we need some basic facts about quadratic forms. A quadratic form is a function of two variables $Q : \mathbb{R}^2 \to \mathbb{R}$ given by
\[
Q(x, y) = ax^2 + 2bxy + cy^2,
\]
where $a$, $b$, and $c$ are constants. $Q$ is said to be definite if $Q(x, x) \neq 0$ whenever $x \neq 0$.

**Exercise 6.** Show that if $ac - b^2 > 0$, then $Q$ is definite, and if $ac - b^2 < 0$, then $Q$ is not definite. (*Hints:* For the first part, suppose that $x \neq 0$, but $Q(x, y) = 0$. Then $ax^2 + 2bxy + cy^2 = 0$, which yields $a + 2b(x/y) + c(x/y)^2 = 0$. Thus the discriminant of this equation must be positive, which will yield a contradiction. The proof of the second part is similar).

Theorem 1 follows from the above exercise.

### 2.3.2 Ratio of areas

In the previous subsection we gave a geometric interpretation for the sign of Gaussian curvature. Here we describe the geometric significance of the magnitude of $K$.

If $V$ is a sufficiently small neighborhood of $p$ in $M$ (where $M$, as always, denotes a regular embedded surface in $\mathbb{R}^3$), then it is easy to show that there exist a patch $(U, X)$ centered at $p$ such that $X(U) = V$. Area of $V$ is then defined as follows:
\[
\text{Area}(V) := \int \int_U \|D_1X \times D_2X\| \, du_1 du_2.
\]

Using the chain rule, one can show that the above definition is independent of the the patch.

**Exercise 7.** Let $V \subset S^2$ be a region bounded in between a pair of great circles meeting each other at an angle of $\alpha$. Show that $\text{Area}(V) = 2\alpha$ (*Hints:* Let $U := [0, \alpha] \times [0, \pi]$ and $X(\theta, \phi) := (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)$. Show that $\|D_1X \times D_2X\| = |\sin \phi|$. Further, note that, after a rotation we may assume that $X(U) = V$. Then an integration will yield the desired result).
Exercise 8. Use the previous exercise to show that the area of a geodesic triangle $T \subset S^2$ (a region bounded by three great circles) is equal to sum of its angles minus $\pi$ \textit{(Hints: Use the picture below: $A + B + C + T = 2\pi$, and $A = 2\alpha - T$, $B = 2\beta - T$, and $C = 2\gamma - T$).}

Let $V_r := B_r(p) \cap M$. Then, if $r$ is sufficiently small, $V(r) \subset X(U)$, and, consequently, $U_r := X^{-1}(V_r)$ is well defined. In particular, we may compute the area of $V_r$ using the patch $(U_r, X)$. In this section we show that

$$|K(p)| = \lim_{r \to 0} \frac{\text{Area}(n(V_r))}{\text{Area}(V_r)}.$$ 

Exercise 9. Recall that the mean value theorem states that $\int_U \int f du_1 du_2 = f(\bar{u}^1, \bar{u}^2) \text{Area}(U)$, for some $(\bar{u}^1, \bar{u}^2) \in U$. Use this theorem to show that

$$\lim_{r \to 0} \frac{\text{Area}(n(V_r))}{\text{Area}(V_r)} = \frac{\|D_1N(0,0) \times D_2N(0,0)\|}{\|D_1X(0,0) \times D_2X(0,0)\|}$$

(Recall that $N := n \circ X$.)

Exercise 10. Prove Lagrange’s identity: for every pair of vectors $v, w \in \mathbb{R}^3$,

$$\|v \times w\|^2 = \det \begin{vmatrix} \langle v, v \rangle & \langle v, w \rangle \\ \langle w, v \rangle & \langle w, w \rangle \end{vmatrix}.$$

Now set $g(u_1, u_2) := \det[g_{ij}(u_1, u_2)]$. Then, by the previous exercise it follows that $\|D_1X(0,0) \times D_2X(0,0)\| = \sqrt{g(0,0)}$. Hence, to complete the proof of the main result of this section it remains to show that

$$\|D_1N(0,0) \times D_2N(0,0)\| = K(p)\sqrt{g(0,0)}.$$
We prove the above formula using two different methods:

**METHOD 1.** Recall that $K(p) := \det(S_p)$, where $S_p := -dn_p: T_pM \to T_pM$ is the shape operator of $M$ at $p$. Also recall that $D_iX(0,0), i = 1, 2$, form a basis for $T_pM$. Let $S_{ij}$ be the coefficients of the matrix representation of $S_p$ with respect to this basis, then

$$S_p(D_iX) = \sum_{j=1}^{2} S_{ij} D_jX.$$

Further, recall that $N := n \circ X$. Thus the chain rule yields:

$$S_p(D_iX) = -dn(D_iX) = -D_i(n \circ X) = -D_iN.$$

**Exercise 11.** Verify the middle step in the above formula, i.e., show that $dn(D_iX) = D_i(n \circ X)$.

From the previous two lines of formulas, it now follows that

$$-D_iN = \sum_{j=1}^{2} S_{ij} D_jX.$$

Taking the inner product of both sides with $D_kN, k = 1, 2$, we get

$$\langle -D_iN, D_kN \rangle = \sum_{j=1}^{2} S_{ij} \langle D_jX, D_kN \rangle.$$

**Exercise 12.** Let $F, G: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a pair of mappings such that $\langle F, G \rangle = 0$. Prove that $\langle D_iF, G \rangle = -\langle F, D_iG \rangle$.

Now recall that $\langle D_jX, N \rangle = 0$. Hence the previous exercise yields:

$$\langle D_jX, D_kN \rangle = -\langle D_{kj}X, N \rangle = -l_{ij}.$$

Combining the previous two lines of formulas, we get: $\langle D_iN, D_kN \rangle = \sum_{k=1}^{2} S_{ij}l_{jk}$; which in matrix notation is equivalent to

$$[\langle D_iN, D_jN \rangle] = [S_{ij}] [l_{ij}].$$

Finally, recall that $\det([\langle D_iN, D_kN \rangle]) = \|D_1N \times D_2N\|^2$, $\det[S_{ij}] = K$, and $\det[l_{ij}] = Kg$. Hence taking the determinant of both sides in the above equation, and then taking the square root yields the desired result.
Next, we discuss the second method for proving that \(\|D_1N \times D_2N\| = K\sqrt{g}\).

**METHOD 2.** Here we work with a special patch which makes the computations easier:

**Exercise 13.** Show that there exist a patch \((U, X)\) centered at \(p\) such that \([g_{ij}(0,0)]\) is the identity matrix. (*Hint: Start with a Monge patch with respect to \(T_pM\)*)

Thus, if we are working with the coordinate patch referred to in the above exercise, \(g(0,0) = 1\), and, consequently, all we need is to prove that \(\|D_1N(0,0) \times D_2N(0,0)\| = K(p)\).

**Exercise 14.** Let \(f: U \subset \mathbb{R}^2 \to \mathbb{S}^2\) be a differentiable mapping. Show that \(\langle D_i f(u_1, u_2), f(u_1, u_2) \rangle = 0\) (*Hints: note that \(\langle f, f \rangle = 1\) and differentiate*).

It follows from the previous exercise that \(\langle D_i N, N \rangle = 0\). Now recall that \(N(0,0) = n \circ X(0,0) = n(p)\). Hence, we may conclude that \(N(0,0) \in T_pM\).

Further recall that \(\{D_1X(0,0), D_2X(0,0)\}\) is now an orthonormal basis for \(T_pM\) (because we have chosen \((U, X)\) so that \([g_{ij}(0,0)]\) is the identity matrix). Consequently,

\[
D_i N = \sum_{k=1}^{2} \langle D_i N, D_k X \rangle D_k X,
\]

where we have omitted the explicit reference to the point \((0,0)\) in the above formula in order to make the notation less cumbersome (it is important to keep in mind, however, that the above is valid only at \((0,0)\)). Taking the inner product of both sides of this equation with \(D_j N(0,0)\) yields:

\[
\langle D_i N, D_j N \rangle = \sum_{k=1}^{2} \langle D_i N, D_k X \rangle \langle D_k X, D_j N \rangle.
\]

Now recall that \(\langle D_i N, D_k X \rangle = -\langle N, D_{ij} X \rangle = -l_{ij}\). Similarly, \(\langle D_k X, D_j N \rangle = -l_{kj}\). Thus, in matrix notation, the above formula is equivalent to the following:

\[
[\langle D_i N, D_j N \rangle] = [l_{ij}]^2
\]

Finally, recall that \(K(p) = \det[l_{ij}(0,0)]/\det[g_{ij}(0,0)] = \det[l_{ij}(0,0)]\). Hence, taking the determinant of both sides of the above equation yields the desired result.
2.3.3 Product of principal curvatures

For every $v \in T_p M$ with $\|v\| = 1$ we define the normal curvature of $M$ at $p$ in the direction of $v$ by

$$k_v(p) := \langle \gamma''(0), n(p) \rangle,$$

where $\gamma: (-\epsilon, \epsilon) \to M$ is a curve with $\gamma(0) = p$ and $\gamma'(0) = v$.

**Exercise 15.** Show that $k_v(p)$ does not depend on $\gamma$.

In particular, by the above exercise, we may take $\gamma$ to be a curve which lies in the intersection of $M$ with a plane which passes through $p$ and is normal to $n(p) \times v$. So, intuitively, $k_v(p)$ is a measure of the curvature of an orthogonal cross section of $M$ at $p$.

Let $UT_pM := \{v \in T_p M \mid \|v\| = 1\}$ denote the unit tangent space of $M$ at $p$. The principal curvatures of $M$ at $p$ are defined as

$$k_1(p) := \min_v k_v(p), \quad \text{and} \quad k_2(p) := \max_v k_v(p),$$

where $v$ ranges over $UT_pM$. Our main aim in this subsection is to show that

$$K(p) = k_1(p)k_2(p).$$

Since $K(p)$ is the determinant of the shape operator $S_p$, to prove the above it suffices to show that $k_1(p)$ and $k_2(p)$ are the eigenvalues of $S_p$.

First, we need to define the second fundamental form of $M$ at $p$. This is a bilinear map $\Pi_p: T_pM \times T_pM \to \mathbb{R}$ defined by

$$\Pi_p(v, w) := \langle S_p(v), w \rangle.$$

We claim that, for all $v \in UT_pM$,

$$k_v(p) = \Pi_p(v, v).$$

The above follows from the following computation

$$\langle S_p(v), v \rangle = -\langle dn_p(v), v \rangle$$

$$= -\langle (n \circ \gamma)'(0), \gamma'(0) \rangle$$

$$= \langle (n \circ \gamma)(0), \gamma''(0) \rangle$$

$$= \langle n(p), \gamma''(0) \rangle.$$
Exercise 16. Verify the passage from the second to the third line in the above computation, i.e., show that
\[-\langle (n \circ \gamma)'(0), \gamma'(0) \rangle = \langle (n \circ \gamma)(0), \gamma''(0) \rangle \]
(Hint: Set \( f(t) := \langle n(\gamma(t)), \gamma'(t) \rangle \), note that \( f(t) = 0 \), and differentiate.)

So we conclude that \( k_1(p) \) are the minimum and maximum of \( \Pi_p(v) \) over \( UT_pM \). Hence, all we need is to show that the extrema of \( \Pi_p \) over \( UT_pM \) coincide with the eigenvalues of \( S_p \).

Exercise 17. Show that \( \Pi_p \) is symmetric, i.e., \( \Pi_p(v, w) = \Pi_p(w, v) \) for all \( v, w \in T_pM \).

By the above exercise, \( S_p \) is a self-adjoint operator, i.e, \( \langle S_p(v), w \rangle = \langle v, S_p(w) \rangle \). Hence \( S_p \) is orthogonally diagonalizable, i.e., there exist orthonormal vectors \( e_i \in T_pM \), \( i = 1, 2 \), such that
\[ S_p(e_i) = \lambda_i e_i. \]

By convention, we suppose that \( \lambda_1 \leq \lambda_2 \). Now note that each \( v \in UT_pM \) may be represented uniquely as \( v = v^1 e_1 + v^2 e_2 \) where \( (v^1)^2 + (v^2)^2 = 1 \). So for each \( v \in UT_pM \) there exists a unique angle \( \theta \in [0, 2\pi) \) such that
\[ v(\theta) := \cos \theta e_1 + \sin \theta e_2; \]

Consequently, bilinearity of \( \Pi_p \) yields
\[ \Pi_p(v(\theta), v(\theta)) = \lambda_1 \cos^2 \theta + \lambda_2 \sin^2 \theta. \]

Exercise 18. Verify the above claim, and show that minimum and maximum values of \( \Pi_p \) are \( \lambda_1 \) and \( \lambda_2 \) respectively. Thus \( k_1(p) = \lambda_1 \), and \( k_2(p) = \lambda_2 \).

The previous exercise completes the proof that \( K(p) = k_1(p)k_2(p) \), and also yields the following formula which was discovered by Euler:
\[ k_v(p) = k_1(p) \cos^2 \theta + k_2(p) \sin^2 \theta. \]

In particular, note that by the above formula there exists always a pair of orthogonal directions where \( k_v(p) \) achieves its maximum and minimum values. These are known as the principal directions of \( M \) at \( p \).