2.6 Gauss’s formulas, and Christoffel Symbols

Let $X : U \to \mathbb{R}^3$ be a proper regular patch for a surface $M$, and set $X_i := D_i X$. Then

$$\{X_1, X_2, N\}$$

may be regarded as a moving bases of frame for $\mathbb{R}^3$ similar to the Frenet Serret frames for curves. We should emphasize, however, two important differences: (i) there is no canonical choice of a moving bases for a surface or a piece of surface ($\{X_1, X_2, N\}$ depends on the choice of the chart $X$); (ii) in general it is not possible to choose a patch $X$ so that $\{X_1, X_2, N\}$ is orthonormal (unless the Gaussian curvature of $M$ vanishes everywhere).

The following equations, the first of which is known as Gauss’s formulas, may be regarded as the analog of Frenet-Serret formulas for surfaces:

$$X_{ij} = \sum_{k=1}^{2} \Gamma^k_{ij} X_k + l_{ij} N,$$

and

$$N_i = -2 \sum_{j=1}^{2} l^j_i X_j.$$

The coefficients $\Gamma^k_{ij}$ are known as the Christoffel symbols, and will be determined below. Recall that $l_{ij}$ are just the coefficients of the second fundamental form. To find out what $l^j_i$ are note that

$$-l_{ik} = -\langle N, X_{ik} \rangle = \langle N_i, X_k \rangle = -\sum_{j=1}^{2} l^j_i \langle X_j, X_k \rangle = -\sum_{j=1}^{2} l^j_i g_{jk}.$$

Thus $(l_{ij}) = (l^j_i)(g_{ij})$. So if we let $(g^{ij}) := (g_{ij})^{-1}$, then $(l^j_i) = (l_{ij})(g^{ij})$, which yields

$$l^j_i = \sum_{k=1}^{2} l_{ik} g^{kj}.$$
**Exercise 1.** What is $\det(H^i_j)$ equal to?

**Exercise 2.** Show that $N_i = -dn(X_i) = S(X_i)$.

Next we compute the Christoffel symbols. To this end note that

$$\langle X_{ij}, X_k \rangle = \sum_{l=1}^{2} \Gamma^l_{ij} \langle X_l, X_k \rangle = \sum_{l=1}^{2} \Gamma^l_{ij} g_{lk},$$

which in matrix notation reads

$$\begin{pmatrix} \langle X_{ij}, X_1 \rangle \\ \langle X_{ij}, X_2 \rangle \end{pmatrix} = \begin{pmatrix} \Gamma^1_{ij} g_{11} + \Gamma^2_{ij} g_{21} \\ \Gamma^1_{ij} g_{12} + \Gamma^2_{ij} g_{22} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} \Gamma^1_{ij} \\ \Gamma^2_{ij} \end{pmatrix}. $$

So

$$\begin{pmatrix} \Gamma^1_{ij} \\ \Gamma^2_{ij} \end{pmatrix} = \begin{pmatrix} g_{11} & g_{21} \\ g_{12} & g_{22} \end{pmatrix}^{-1} \begin{pmatrix} \langle X_{ij}, X_1 \rangle \\ \langle X_{ij}, X_2 \rangle \end{pmatrix} = \begin{pmatrix} g^{11} & g^{21} \\ g^{12} & g^{22} \end{pmatrix} \begin{pmatrix} \langle X_{ij}, X_1 \rangle \\ \langle X_{ij}, X_2 \rangle \end{pmatrix},$$

which yields

$$\Gamma^k_{ij} = \sum_{l=1}^{2} \langle X_{ij}, X_l \rangle g^{lk}. $$

In particular, $\Gamma^k_{ij} = \Gamma^k_{ji}$. Next note that

$$\begin{align*}
(g_{ij})_k &= \langle X_{ik}, X_j \rangle + \langle X_i, X_{jk} \rangle, \\
(g_{jk})_i &= \langle X_{ji}, X_k \rangle + \langle X_j, X_{ki} \rangle, \\
(g_{ki})_j &= \langle X_{kj}, X_i \rangle + \langle X_k, X_{ij} \rangle. 
\end{align*}$$

Thus

$$\langle X_{ij}, X_k \rangle = \frac{1}{2} ((g_{ki})_j + (g_{jk})_i - (g_{ij})_k).$$

So we conclude that

$$\Gamma^k_{ij} = \sum_{l=1}^{2} \frac{1}{2} ((g_{li})_j + (g_{lj})_i - (g_{ij})_l) g^{lk}. $$

Note that the last equation shows that $\Gamma^k_{ij}$ are *intrinsic quantities*, i.e., they depend only on $g_{ij}$ (and derivatives of $g_{ij}$), and so are preserved under isometries.

**Exercise 3.** Compute the Christoffel symbols of a surface of revolution.
2.7 The Gauss and Codazzi-Mainardi Equations, Riemann Curvature Tensor, and a Second Proof of Gauss’s Theorema Egregium

Here we shall derive some relations between $l_{ij}$ and $g_{ij}$. Our point of departure is the simple observation that if $X : U \to \mathbb{R}^3$ is a $C^3$ regular patch, then, since partial derivatives commute,

$$X_{ijk} = X_{ikj}.$$ 

Note that

$$X_{ijk} = \left( \sum_{l=1}^{2} \Gamma^l_{ij} X_l + l_{ij} N \right)_k$$

$$= \sum_{l=1}^{2} (\Gamma^l_{ij})_k X_l + \sum_{l=1}^{2} \Gamma^l_{ij} X_{lk} + (l_{ij})_k N + l_{ij} N_k$$

$$= \sum_{l=1}^{2} (\Gamma^l_{ij})_k X_l + \sum_{l=1}^{2} \Gamma^l_{ij} \left( \sum_{m=1}^{2} \Gamma^m_{lk} X_m + l_{lk} N \right) + (l_{ij})_k N - l_{ij} \sum_{l=1}^{2} l_{lk} X_l$$

$$= \sum_{l=1}^{2} (\Gamma^l_{ij})_k X_l + \sum_{l=1}^{2} \sum_{m=1}^{2} \Gamma^l_{ij} \sum_{l=1}^{2} \Gamma^m_{lk} X_m + \sum_{l=1}^{2} \Gamma^l_{ij} l_{lk} N + (l_{ij})_k N - \sum_{l=1}^{2} l_{ij} l_{lk} X_l$$

$$= \sum_{l=1}^{2} \left( (\Gamma^l_{ij})_k + \sum_{p=1}^{2} \Gamma^p_{ij} \Gamma^l_{pk} - l_{ij} l_{lk} \right)_k X_l + \left( \sum_{l=1}^{2} \Gamma^l_{ij} l_{lk} + (l_{ij})_k \right) N.$$ 

Switching $k$ and $j$ yields,

$$X_{ikj} = \sum_{l=1}^{2} \left( (\Gamma^l_{ik})_j + \sum_{p=1}^{2} \Gamma^p_{ik} \Gamma^l_{pj} - l_{ik} l_{jk} \right)_k X_l + \left( \sum_{l=1}^{2} \Gamma^l_{ik} l_{lj} + (l_{ik})_j \right) N.$$ 

Setting the normal and tangential components of the last two equations equal to each other we obtain

$$(\Gamma^l_{ij})_k + \sum_{p=1}^{2} \Gamma^p_{ij} \Gamma^l_{pk} - l_{ij} l_{lk} = (\Gamma^l_{ik})_j + \sum_{p=1}^{2} \Gamma^p_{ik} \Gamma^l_{pj} - l_{ik} l_{jk},$$

$$\sum_{l=1}^{2} \Gamma^l_{ij} l_{lk} + (l_{ij})_k = \sum_{l=1}^{2} \Gamma^l_{ik} l_{lj} + (l_{ik})_j.$$
These equations may be rewritten as

\[(\Gamma^l_{ik})_j - (\Gamma^l_{ij})_k + \sum_{p=1}^{2} (\Gamma^p_{ik} \Gamma^l_{pj} - \Gamma^p_{ij} \Gamma^l_{pk}) = l_{ik} l^l_{jk} - l_{ij} l^l_{lk}, \quad \text{(Gauss)}\]

\[\sum_{l=1}^{2} (\Gamma^l_{ik} l_{lj} - \Gamma^l_{ij} l_{lk}) = (l_{ij})_k - (l_{ik})_j, \quad \text{(Codazzi-Mainardi)}\]

and are known as the Gauss’s equations and the Codazzi-Mainardi equations respectively. If we define the Riemann curvature tensor as

\[R^l_{ijk} := (\Gamma^l_{ik})_j - (\Gamma^l_{ij})_k + \sum_{p=1}^{2} (\Gamma^p_{ik} \Gamma^l_{pj} - \Gamma^p_{ij} \Gamma^l_{pk}),\]

then Gauss’s equation may be rewritten as

\[R^l_{ijk} = l_{ik} l^l_{jk} - l_{ij} l^l_{lk}.\]

Now note that

\[\sum_{l=1}^{2} R^l_{ijk} g_{tm} = l_{ik} \sum_{l=1}^{2} l^l_{jm} - l_{ij} \sum_{l=1}^{2} l^l_{km} = l_{ik} l_{jm} - l_{ij} l_{km}.\]

In particular, if \(i = k = 1\) and \(j = m = 2\), then

\[\sum_{l=1}^{2} R^l_{121} g_{22} = l_{11} l_{22} - l_{12} l_{21} = \det(l_{ij}) = K \det(g_{ij}).\]

So it follows that

\[K = \frac{R^1_{121} g_{12} + R^2_{121} g_{22}}{\det(g_{ij})},\]

which shows that \(K\) is intrinsic and gives another proof of Gauss’s Theorema Egregium.

**Exercise 4.** Show that if \(M = \mathbf{R}^2\), hen \(R^l_{ijk} = 0\) for all \(1 \leq i, l, j, k \leq 2\) both intrinsically and extrinsically.

**Exercise 5.** Show that (i) \(R^l_{ijk} = -R^l_{ikj}\), hence \(R^l_{ijj} = 0\), and (ii) \(R^l_{ijk} + R^l_{jki} + R^l_{kji} \equiv 0\).

**Exercise 6.** Compute the Riemann curvature tensor for \(S^2\) both intrinsically and extrinsically.
2.8 Fundamental Theorem of Surfaces

In the previous section we showed that if \( g_{ij} \) and \( l_{ij} \) are the coefficients of the first and second fundamental form of a patch \( X: U \to M \), then they must satisfy the Gauss and Codazzi-Mainardi equations. These conditions turn out to be not only necessary but also sufficient in the following sense.

**Theorem 7 (Fundamental Theorem of Surfaces).** Let \( U \subset \mathbb{R}^2 \) be an open neighborhood of the origin \((0,0)\), and \( g_{ij}: U \to \mathbb{R} \), \( l_{ij}: U \to \mathbb{R} \) be differentiable functions for \( i, j = 1, 2 \). Suppose that \( g_{ij} = g_{ji} \), \( l_{ij} = l_{ji} \), \( g_{11} > 0 \), \( g_{22} > 0 \) and \( \det(g_{ij}) > 0 \). Further suppose that \( g_{ij} \) and \( l_{ij} \) satisfy the Gauss and Codazzi-Mainardi equations. Then there exists an open set \( V \subset U \), with \((0,0) \in V\) and a regular patch \( X: V \to \mathbb{R} \) with \( g_{ij} \) and \( l_{ij} \) as its first and second fundamental forms respectively. Further, if \( Y: V \to \mathbb{R}^3 \) is another regular patch with first and second fundamental forms \( g_{ij} \) and \( l_{ij} \), then \( Y \) differs from \( X \) by a rigid motion.