2.14 Applications of the Gauss-Bonnet theorem

We talked about the Gauss-Bonnet theorem in class, and you may find the statement and prove of it in Gray or do Carmo as well. The following are all simple consequences of the Gauss-Bonnet theorem:

Exercise 1. Show that the sum of the angles in a triangle is $\pi$.

Exercise 2. Show that the total geodesic curvature of a simple closed planar curve is $2\pi$.

Exercise 3. Show that the Gaussian curvature of a surface which is homeomorphic to the torus must always be equal to zero at some point.

Exercise 4. Show that a simple closed curve with total geodesic curvature zero on a sphere bisects the area of the sphere.

Exercise 5. Show that there exists at most one closed geodesic on a cylinder with negative curvature.

Exercise 6. Show that the area of a geodesic polygon with $k$ vertices on a sphere of radius 1 is equal to the sum of its angles minus $(k - 2)\pi$.

Exercise 7. Let $p$ be a point of a surface $M$, $T$ be a geodesic triangle which contains $p$, and $\alpha$, $\beta$, $\gamma$ be the angles of $T$. Show that

$$K(p) = \lim_{T \to p} \frac{\alpha + \beta + \gamma - \pi}{\text{Area}(T)}.$$

In particular, note that the above proves Gauss’s Theorema Egregium.

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Exercise 8. Show that the sum of the angles of a geodesic triangle on a surface of positive curvature is more than $\pi$, and on a surface of negative curvature is less than $\pi$.

Exercise 9. Show that on a simply connected surface of negative curvature two geodesics emanating from the same point will never meet.

Exercise 10. Let $M$ be a surface homeomorphic to a sphere in $\mathbf{R}^3$, and let $\Gamma \subset M$ be a closed geodesic. Show that each of the two regions bounded by $\Gamma$ have equal areas under the Gauss map.

Exercise 11. Compute the area of the pseudo-sphere, i.e. the surface of revolution obtained by rotating a tractrix.