2 Surfaces

2.1 Definition of a regular embedded surface

An \( n \)-dimensional open ball of radius \( r \) centered at \( p \) is defined by

\[
B^n_r(p) := \{ x \in \mathbb{R}^n \mid \text{dist}(x, p) < r \}.
\]

We say a subset \( U \subset \mathbb{R}^n \) is open if for each \( p \) in \( U \) there exists an \( \epsilon > 0 \) such that \( B^n_\epsilon(p) \subset U \). Let \( A \subset \mathbb{R}^n \) be an arbitrary subset, and \( U \subset A \). We say that \( U \) is open in \( A \) if there exists an open set \( V \subset \mathbb{R}^n \) such that \( U = A \cap V \). A mapping \( f: A \to B \) between arbitrary subsets of \( \mathbb{R}^n \) is said to be continuous if for every open set \( U \subset B \), \( f^{-1}(U) \) is open in \( A \). Intuitively, we may think of a continuous map as one which sends nearby points to nearby points:

**Exercise 1.** Let \( A, B \subset \mathbb{R}^n \) be arbitrary subsets, \( f: A \to B \) be a continuous map, and \( p \in A \). Show that for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that whenever \( \text{dist}(x, p) < \delta \), then \( \text{dist}(f(x), f(p)) < \epsilon \).

Two subsets \( A, B \subset \mathbb{R}^n \) are said to be homeomorphic, or topologically equivalent, if there exists a mapping \( f: A \to B \) such that \( f \) is one-to-one, onto, continuous, and has a continuous inverse. Such a mapping is called a homeomorphism. We say a subset \( M \subset \mathbb{R}^3 \) is an embedded surface if every point in \( M \) has an open neighborhood in \( M \) which is homeomorphic to an open subset of \( \mathbb{R}^2 \).

**Exercise 2. (Stereographic projection)** Show that the standard sphere \( S^2 := \{ p \in \mathbb{R}^3 \mid \|p\| = 1 \} \) is an embedded surface (Hint: Show that the stereographic projection \( \pi_+ \) form the north pole gives a homeomorphism between \( \mathbb{R}^2 \) and \( S^2 - (0,0,1) \). Similarly, the stereographic projection \( \pi_- \)
from the south pole gives a homeomorphism between \( \mathbb{R}^2 \) and \( S^2 - (0, 0, -1) \); 
\[ \pi_+(x, y, z) := \left( \frac{x}{1-z}, \frac{y}{1-z}, 0 \right), \quad \pi_-(x, y, z) := \left( \frac{x}{z-1}, \frac{y}{z-1}, 0 \right) \].

**Exercise 3. (Surfaces as graphs)** Let \( U \subset \mathbb{R}^2 \) be an open subset and \( f: U \to \mathbb{R} \) be a continuous map. Then 

\[
\text{graph}(f) := \{ (x, y, f(x, y)) \mid (x, y) \in U \}
\]
is a surface. (*Hint:* Show that the orthogonal projection \( \pi(x, y, z) := (x, y) \) gives the desired homeomorphism).

Note that by the above exercise the cone given by \( z = \sqrt{x^2 + y^2} \), and the troughlike surface \( z = |x| \) are examples of embedded surfaces. These surfaces, however, are not “regular”, as we will define below. From the point of view of differential geometry it is desirable that a surface be without sharp corners or vertices.

Let \( U \subset \mathbb{R}^n \) be open, and \( f: U \to \mathbb{R}^m \) be a map. Note that \( f \) may be regarded as a list of \( m \) functions of \( n \) variables: \( f(p) = (f^1(p), \ldots, f^m(p)) \), \( f^i(p) = f^i(p^1, \ldots, p^n) \). The first order partial derivatives of \( f \) are given by 

\[
D_jf^i(p) := \lim_{h \to 0} \frac{f^i(p^1, \ldots, p^j + h, \ldots, p^n) - f^i(p^1, \ldots, p^j, \ldots, p^n)}{h}.
\]

If all the functions \( D_jf^i: U \to \mathbb{R} \) exist and are continuous, then we say that \( f \) is differentiable \((C^1)\). We say that \( f \) is smooth \((C^\infty)\) if the partial derivatives of \( f \) of all order exist and are continuous. These are defined by 

\[
D_{j_1, j_2, \ldots, j_k}f^i := D_{j_1}(D_{j_2}(\cdots(D_{j_k}f^i)\cdots)).
\]

Let \( f: U \subset \mathbb{R}^n \to \mathbb{R}^m \) be a differentiable map, and \( p \in U \). Then the Jacobian of \( f \) at \( p \) is an \( m \times n \) matrix defined by 

\[
J_p(f) := \begin{pmatrix}
D_1f^1(p) & \cdots & D_nf^1(p) \\
\vdots & \cdots & \vdots \\
D_1f^m(p) & \cdots & D_nf^m(p)
\end{pmatrix}.
\]

We say that \( p \) is a regular point of \( f \) if the rank of \( J_p(f) \) is equal to \( n \). If \( f \) is regular at all points \( p \in U \), then we say that \( f \) is regular.

**Exercise 4 (Monge Patch).** Let \( f: U \subset \mathbb{R}^2 \to \mathbb{R} \) be a differentiable map. Show that the mapping \( X: U \to \mathbb{R}^3 \), defined by \( X(u^1, u^2) := (u^1, u^2, f(u^1, u^2)) \) is regular (the pair \((X, U)\) is called a Monge Patch).
If \( f \) is a differentiable function, then we define,
\[
D_i f(p) := (D_1 f^1(p), \ldots, D_i f^n(p)).
\]

Exercise 5. Show that \( f : U \subset \mathbb{R}^2 \to \mathbb{R}^3 \) is regular at \( p \) if and only if
\[
\| D_1 f(p) \times D_2 f(p) \| \neq 0.
\]

Let \( f : U \subset \mathbb{R}^n \to \mathbb{R}^m \) be a differentiable map and \( p \in U \). Then the differential of \( f \) at \( p \) is a mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) defined by
\[
df_p(x) := \lim_{t \to 0} \frac{f(p+tx) - f(p)}{t}.
\]

Exercise 6. Show that (i)
\[
df_p(x) = J_p(f) \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}.
\]

Conclude then that (ii) \( df \) is a linear map, and (iii) \( p \) is a regular value of \( f \) if and only if \( df \) is one-to-one. Further, (iv) show that if \( f \) is a linear map, then \( df_p(x) = f(x) \), and (v) \( J_p(f) \) coincides with the matrix representation of \( f \) with respect to the standard basis.

By a regular patch we mean a pair \((U, X)\) where \( U \subset \mathbb{R}^2 \) is open and \( X : U \to \mathbb{R}^3 \) is a one-to-one, smooth, and regular mapping. Furthermore, we say that the patch is proper if \( X^{-1} \) is continuous. We say a subset \( M \subset \mathbb{R}^3 \) is a regular embedded surface, if for each point \( p \in M \) there exists a proper regular patch \((U, X)\) and an open set \( V \subset \mathbb{R}^3 \) such that \( X(U) = M \cap V \). The pair \((U, X)\) is called a local parameterization for \( M \) at \( p \).

Exercise 7. Let \( f : U \subset \mathbb{R}^2 \to \mathbb{R} \) be a smooth map. Show that \( \text{graph}(f) \) is a regular embedded surface, see Exercise 4.

Exercise 8. Show that \( S^2 \) is a regular embedded surface (Hint: (Method 1) Let \( p \in S^2 \). Then \( p^1, p^2, \) and \( p^3 \) cannot vanish simultaneously. Suppose, for instance, that \( p^3 \neq 0 \). Then, we may set \( U := \{ u \in \mathbb{R}^2 \mid \|u\| < 1 \} \), and let \( X(u^1, u^2) := (u^1, u^2, \pm \sqrt{1 - (u^1)^2 - (u^2)^2}) \) depending on whether \( p^3 \) is positive or negative. The other cases involving \( p^1 \) and \( p^2 \) may be treated similarly. (Method 2) Write the inverse of the stereographic projection, see Exercise 2, and show that it is a regular map).
The following exercise shows that smoothness of a patch is not sufficient to ensure that the corresponding surface is without singularities (sharp edges or corners). Thus the regularity condition imposed in the definition of a regular embedded surface is not superfluous.

**Exercise 9.** Let \( M \subset \mathbb{R}^3 \) be the graph of the function \( f(x, y) = |x| \). Sketch this surface, and show that there exists a smooth one-to-one map \( X: \mathbb{R}^2 \to \mathbb{R}^3 \) such that \( X(\mathbb{R}^2) = M \) (Hint: Let \( X(x, y) := (e^{-1/x^2}, y, e^{-1/x^2}) \), if \( x > 0 \); \( X(x, y) := (-e^{-1/x^2}, y, e^{-1/x^2}) \), if \( x < 0 \); and, \( X(x, y) := (0, 0, 0) \), if \( x = 0 \)).

The following exercise demonstrates the significance of the requirement in the definition of a regular embedded surface that \( X^{-1} \) be continuous.

**Exercise 10.** Let \( U := \{(u, v) \in \mathbb{R}^2 \mid -\pi < u < \pi, \ 0 < v < 1\} \), define \( X: U \to \mathbb{R}^3 \) by \( X(u, v) := (\sin(u), \sin(2u), v) \), and set \( M := X(U) \). Sketch \( M \) and show that \( X \) is smooth, one-to-one, and regular, but \( X^{-1} \) is not continuous.

**Exercise 11 (Surfaces of Revolution).** Let \( \alpha: I \to \mathbb{R}^2, \alpha(t) = (x(t), y(t)) \), be a regular simple closed curve. Show that the image of \( X: I \times \mathbb{R} \to \mathbb{R}^3 \) given by
\[
X(t, \theta) := \left( x(t) \cos \theta, x(t) \sin \theta, y(t) \right),
\]
is a regular embedded surface.