2.5 The inverse function theorem

Recall that if \( f: M \to N \) is a diffeomorphism, then \( df_p \) is nonsingular at all \( p \in M \) (by the chain rule and the observation that \( f \circ f^{-1} \) is the identity function on \( M \)). The main aim of this section is to prove a converse of this phenomenon:

**Theorem 1 (The Inverse Function Theorem).** Let \( f: M \to N \) be a smooth map, and \( \dim(M) = \dim(N) \). Suppose that \( df_p \) is nonsingular at some \( p \in M \). Then \( f \) is a local diffeomorphism at \( p \), i.e., there exists an open neighborhood \( U \) of \( p \) such that

1. \( f \) is one-to-one on \( U \).
2. \( f(U) \) is open in \( N \).
3. \( f^{-1}: f(U) \to U \) is smooth.

In particular, \( d(f^{-1})_{f(p)} = (df_p)^{-1} \).

A simple fact which is applied a number of times in the proof of the above theorem is

**Lemma 2.** Let \( f: M \to N \), and \( g: N \to L \) be diffeomorphisms, and set \( h := g \circ f \). If any two of the mappings \( f \), \( g \), \( h \) are diffeomorphisms, then so is the third.

In particular, the above lemma implies

**Proposition 3.** If Theorem 1 is true in the case of \( M = \mathbb{R}^n = N \), then, it is true in general.
Proof. Suppose that Theorem 1 is true in the case that $M = \mathbb{R}^n = N$, and let $f: M \to N$ be a smooth map with $df_p$ nonsingular at some $p \in M$. By definition, there exist local charts $(U, \phi)$ of $M$ and $(V, \psi)$ of $N$, centered at $p$ and $f(p)$ respectively, such that $\tilde{f} := \phi^{-1} \circ f \circ \psi$ is smooth. Since $\phi$ and $\psi$ are diffeomorphisms, $d\phi_p$ and $d\psi_{f(p)}$ are nonsingular. Consequently, by the chain rule, $df_0$ is nonsingular, and is thus a local diffeomorphism. More explicitly, there exists open neighborhoods $A$ and $B$ of the origin $o$ of $\mathbb{R}^n$ such that $\tilde{f}: A \to B$ is a diffeomorphism. Since $\phi: \phi^{-1}(A) \to A$ is also a diffeomorphism, it follows that $\phi \circ \tilde{f}: \phi^{-1}(A) \to B$ is a diffeomorphism. But $\phi \circ \tilde{f} = f \circ \psi$. So $f \circ \psi: \phi^{-1}(A) \to B$ is a diffeomorphism. Finally, since $\psi: \psi^{-1}(B) \to B$ is a diffeomorphism, it follows, by the above lemma, that $f: \phi^{-1}(A) \to \psi^{-1}(B)$ is a diffeomorphism.

So it remains to prove Theorem 1 in the case that $M = \mathbb{R}^n = N$. To this end we need the following fact. Recall that a metric space is said to be complete provided that every Cauchy sequence of that space converges.

**Lemma 4 (The contraction Lemma).** Let $(X, d)$ be a complete metric space, and $0 \leq \lambda < 1$. Suppose that there exists mapping $f: X \to X$ such that $d(f(x_1), (x_2)) \leq \lambda d(x_1, x_2)$, for all $x_1, x_2 \in X$. Then there exists a unique point $x \in X$ such that $f(x) = x$.

**Proof.** Pick a point $x_0 \in X$ and set $x_n := f^n(x)$, for $n \geq 1$. We claim that $\{x_n\}$ is a Cauchy sequence. To this end note that

$$d(x_n, x_{n+m}) = d(f^n(x_0), f^m(x_m)) \leq \lambda^n d(x_0, x_m).$$

Further, by the triangle inequality

$$d(x_0, x_m) \leq d(x_0, x_1) + d(x_1, x_2) + \cdots + d(x_{m-1}, x_m) \leq (1 + \lambda + \lambda^2 + \cdots + \lambda^m)d(x_0, x_1) \leq \frac{1}{1 - \lambda}d(x_0, x_1).$$

So, setting $K := d(x_0, x_1)/(1 - \lambda)$, we have

$$d(x_n, x_{n+m}) \leq \lambda^n K.$$

Since $K$ does not depend on $m$ or $n$, the last inequality shows that $\{x_n\}$ is a Cauchy sequence, and therefore, since $X$ is complete, it has a limit point, say $x_\infty$. Now note that, since $d: X \times X \to \mathbb{R}$ is continuous (why?),

$$d(x_\infty, f(x_\infty)) = \lim_{n \to \infty} d(x_n, f(x_n)) = 0.$$
Thus $X_\infty$ is a fixed point of $f$. Finally, note that if $a$ and $b$ are fixed points of $f$, then
\[ d(a, b) = d(f(a), f(b)) \leq \lambda d(a, b), \]
which, since $\lambda < 1$, implies that $d(a, b) = 0$. So $f$ has a unique fixed point. \qed

**Exercise 5.** Does the previous lemma remain valid if the condition that $d(f(x_1), (x_2)) \leq \lambda d(x_1, x_2)$ is weakened to $d(f(x_1), (x_2)) \leq d(x_1, x_2)$?

Next we recall

**Lemma 6 (The mean value theorem).** Let $f: \mathbb{R}^n \to \mathbb{R}$ be a $C^1$ function. Then for every $p, q \in \mathbb{R}^n$ there exists a point $s$ on the line segment connecting $p$ and $q$ such that
\[
 f(p) - f(q) = Df(s)(p - q) = \sum_{i=1}^{n} D_i f(s_i)(p^i - q^i).
\]

**Exercise 7.** Prove the last lemma by using the mean value theorem for functions of one variable and the chain rule. (Hint: Parametrize the segment joining $p$ and $q$ by $tq + (1 - t)p$, $0 \leq t \leq 1$).

The above lemma implies:

**Proposition 8.** Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a $C^1$ function, $U$ be a convex open neighborhood of $o$ in $\mathbb{R}^n$, and set
\[
 K := \sup \left\{ \left| D_{j}f^i(p) \right| \middle| 1 \leq i \leq m, 1 \leq j \leq n, p \in U \right\}
\]
Then, for every $p, q \in U$,
\[
 \|f(p) - f(q)\| \leq \sqrt{mn} K \|p - q\|
\]

**Proof.** First note that
\[
 \|f(p) - f(q)\|^2 = \sum_{i=1}^{m} (f^i(p) - f^i(q))^2.
\]
Secondly, by the mean value theorem (Lemma 6), there exists, for every \( i \) a point \( s_i \) on the line segment connecting \( p \) and \( q \) such that

\[
f^i(p) - f^i(q) = Df^i(s_i)(p - q) = \sum_{j=1}^{n} D_j f^i(s_j)(p^j - q^j).
\]

Since \( U \) is convex, \( s_i \in U \), and, therefore, by the Cauchy-Schwartz inequality

\[
|f^i(p) - f^i(q)| \leq \sqrt{\sum_{j=1}^{n} D_j f^i(s_j)^2} \sqrt{\sum_{j=1}^{n} (p^j - q^j)^2} \leq \sqrt{n}K\|p - q\|.
\]

So we conclude that

\[
\|f(p) - f(q)\|^2 \leq mnK^2\|p - q\|^2.
\]

\[\Box\]

Finally, we recall the following basic fact

**Lemma 9.** Let \( f: \mathbb{R}^n \to \mathbb{R}^m \), and \( p \in \mathbb{R}^n \). Suppose there exists a linear transformation \( A: \mathbb{R}^n \to \mathbb{R}^m \) such that

\[
f(x) - f(p) = A(p - x) + r(x, p)
\]

where \( r: \mathbb{R}^2 \to \mathbb{R} \) is a function satisfying

\[
\lim_{x \to p} \frac{r(x, p)}{\|x - p\|} = 0.
\]

Then all the partial derivatives of \( f \) exist at \( p \), and \( A \) is given by the jacobian matrix \( Df(p) := (D_1 f(p), \ldots, D_n f(p)) \) whose columns are the partial derivatives of \( f \). In particular, \( A \) is unique. Conversely, if all the partial derivative \( D_i f(p) \) exist, then \( A := Df(p) \) satisfies the above equation.

**Proof.** Let \( e_1, \ldots, e_n \) be the standard basis for \( \mathbb{R}^n \). Then

\[
D_i f(p) = \lim_{t \to 0} \frac{f(p + te_i) - f(p)}{t} = \lim_{t \to 0} \frac{A(te_i) + r(p + te_i, p)}{t} = A(e_i).
\]

Thus all the partial derivatives of \( f \) exist at \( p \), and \( D_i f(p) \) coincides with the \( i^{th} \) column of (the matrix representation) of \( A \). In particular, \( A = Df(p) \) and therefore \( A \) is unique.
Conversely, suppose that all the partial derivatives \( D_i f(p) \) exist and set

\[
r(x, p) := f(x) - f(p) - D f(p)(p - x).
\]

By the mean value theorem,

\[
r(x, p) = (D f(s) - D f(p))(p - x)
\]

for some \( s \) on the line segment joining \( p \) and \( s \). Thus it follows that

\[
\lim_{x \to p} \frac{r(x, p)}{\|x - p\|} = \lim_{x \to p} \left( D f(s) - D f(p) \right) \left( \frac{p - x}{\|p - x\|} \right) = 0,
\]

as desired. \( \Box \)

Now we are finally ready to prove the main result of this section.

**Proof of Theorem 1.** By 3 we may assume that \( M = \mathbb{R}^n = N \). Further, after replacing \( f(x) \) with \( (D f(p))^{-1} f(x - p) - f(p) \) we may assume, via Lemma 2, that

\[
p = o, \quad f(o) = o, \quad \text{and} \quad D f(o) = I,\]

where \( I \) denotes the identity matrix. Now define \( g: \mathbb{R}^n \to \mathbb{R}^n \) by

\[
g(x) = x - f(x).
\]

Then \( g(o) = o \), and \( D g(o) = 0 \). Thus, by Proposition 8, there exists \( r > 0 \) such that for all \( x_1, x_2 \in B_r(o) \), the closed ball of radius \( r \) centered at \( o \),

\[
\|g(x_1) - g(x_2)\| \leq \frac{1}{2} \|x_1 - x_2\|.
\]

In particular, \( \|g(x)\| = \|g(x) - g(o)\| \leq \|x\|/2 \). So \( g(B_r(o)) \subset B_{r/2}(o) \). Now, for every \( y \in B_{r/2}(o) \) and \( x \in B_r(o) \) define

\[
T_y(x) := y + g(x) = y + x - f(x).
\]

Then, by the triangle inequality, \( \|T_y(x)\| \leq r \). Thus \( T_y: B_r(o) \to B_r(o) \). Further note that

\[
T_y(x) = x \iff y = f(x).
\]
in particular, $T_y$ has a unique fixed point on $B_r(o)$ if and only if $f$ is one-to-one on $B_r(o)$. But

$$\|T_y(x_1) - T_y(x_2)\| = \|g(x_1) - g(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\|.$$ 

Thus by Lemma 4, $T_y$ does indeed have a unique fixed point, and we conclude that $f$ is one-to-one on $B_r(o)$. In particular, we let $U$ be the interior of $B_r(o)$.

Next we show that $f(U)$ is open. To this end it suffices to prove that $f^{-1}: f(B_r(o)) \to B_r(o)$ is continuous. To see this note that, by the definition of $g$ and the triangle inequality,

$$\|g(x_1) - g(x_2)\| = \|(x_1 - x_2) - (f(x_1) - f(x_2))\| \geq \|x_1 - x_2\| - \|f(x_1) - f(x_2)\|.$$ 

Thus,

$$\|f(x_1) - f(x_2)\| \geq \|x_1 - x_2\| - \|g(x_1) - g(x_2)\| = \frac{1}{2}\|x_1 - x_2\|,$$

which in turn implies

$$\|y_1 - y_2\| \geq \frac{1}{2}\|f^{-1}(y_1) - f^{-1}(y_2)\|.$$ 

So $f^{-1}$ is continuous.

It remains to show that $f^{-1}$ is smooth on $f(U)$. To this end, note that by Lemma 9, for every $p \in U$,

$$f(x) - f(p) = Df(p)(x - p) + r(x, p).$$ 

Now multiply both sides of the above equality by $A := (Df(p))^{-1}$, and set $y := f(x)$, $q := f(p)$. Then

$$A(y - q) = f^{-1}(y) - f^{-1}(q) + Ar(f^{-1}(y), f^{-1}(q)),$$

which we may rewrite as

$$f^{-1}(y) - f^{-1}(q) = A(y - q) + \overline{r}(y, q),$$

where

$$\overline{r}(y, q) := Ar(f^{-1}(y), f^{-1}(q)).$$
Finally note that
\[
\lim_{y \to q} \frac{r(y, q)}{\|y - q\|} = A \lim_{y \to q} \frac{r(f^{-1}(y), f^{-1}(q))}{\|y - q\|} \leq 2 A \lim_{y \to q} \frac{r(f^{-1}(y), f^{-1}(q))}{\|f^{-1}(y) - f^{-1}(q)\|} = 0.
\]
Thus, again by Lemma 9, \(f^{-1}\) is differentiable at all \(p \in U\) and

\[
D(f^{-1})(p) = \left( Df(f^{-1}(p)) \right)^{-1}.
\]

Since the right hand side of the above equation is a continuous function of \(p\) (because \(f\) is \(C^1\) and \(f^{-1}\) is continuous), it follows that \(f^{-1}\) is \(C^1\). But if \(f\) is \(C^r\), then the right hand side of the above equation is \(C^r\) (since \(Df\) is \(C^\infty\) everywhere), which in turn yields that \(f^{-1}\) is \(C^{r+1}\). So, by induction, \(f^{-1}\) is \(C^\infty\).

Exercise 10. Give a simpler proof of the inverse function theorem for the special case of mappings \(f : \mathbb{R} \to \mathbb{R}\).

2.6 The rank theorem

The inverse function theorem we proved in the last section yields the following more general result:

Theorem 11 (The rank theorem). Let \(f : M \to N\) be a smooth map, and suppose that \(\text{rank}(df_p) = k\) for all \(p \in M\), then, for each \(p \in M\), there exists local charts \((U, \phi)\) and \((V, \psi)\) of \(M\) and \(N\) centered at \(p\) and \(f(p)\) respectively such that

\[
\psi \circ f \circ \phi^{-1}(x_1, \ldots, x_n) = (x_1, \ldots, x_k, 0, \ldots, 0).
\]

Exercise 12. Show that to prove the above theorem it suffices to consider the case \(M = \mathbb{R}^n\) and \(N = \mathbb{R}^m\). Furthermore, show that we may assume that \(p = o\), \(f(o) = o\), and the \(k \times k\) matrix in the upper left corner of the jacobian matrix \(Df(o)\) is nonsingular.

Proof. Suppose that the conditions of the previous exercise hold. Define \(\phi : \mathbb{R}^n \to \mathbb{R}^n\) by

\[
\phi(x) := (f^1(x), \ldots, f^k(x), x^{k+1}, \ldots, x^n).
\]
Then
\[ D\phi(o) = \begin{pmatrix} \frac{\partial(f_1, \ldots, f_k)}{(x^1, \ldots, x^k)}(o) & 0 \\ 0 & I_{n-k} \end{pmatrix}. \]

Thus \( D\phi(o) \) is nonsingular. So, by the inverse function theorem, \( \phi \) is a local diffeomorphism at \( o \). In particular \( \phi^{-1} \) is well defined on some open neighborhood \( U \) of \( o \). Let \( \pi_i : \mathbb{R}^k \to \mathbb{R} \) be the projection onto the \( i \)th coordinate. Then, for \( 1 \leq i \leq k \), \( \pi_i \circ \phi = f^i \). Consequently, \( f^i \circ \phi^{-1} = \pi_i \). Thus, if we set \( \tilde{f}^i := f^i \circ \phi^{-1} \), for \( k+1 \leq i \leq m \), then
\[ f \circ \phi^{-1}(x) = (x^1, \ldots, x^k, \tilde{f}^{k+1}(x), \ldots, \tilde{f}^m(x)) \]
for all \( x \in U \). Next note that
\[ D(f \circ \phi^{-1})(o) = \begin{pmatrix} I_k & 0 \\ \ast & \frac{\partial(\tilde{f}^{k+1}, \ldots, \tilde{f}^m)}{(x^{k+1}, \ldots, x^n)}(o) \end{pmatrix}. \]

On the other hand, \( D(f \circ \phi^{-1})(o) = D(f)(p) \circ D(\phi^{-1})(o) \). Thus
\[ \text{rank}(D(f \circ \phi^{-1})(o)) = \text{rank}(D(f)(p)) = k, \]
because \( D(\phi^{-1}) = D(\phi)^{-1} \) is nonsingular. The last two equalities imply that
\[ \frac{\partial(\tilde{f}^{k+1}, \ldots, \tilde{f}^m)}{(x^{k+1}, \ldots, x^n)}(o) = 0, \]
where 0 here denotes the matrix all of whose entries is zero. So we conclude that the functions \( \tilde{f}^{k+1}, \ldots, \tilde{f}^m \) do not depend on \( x^{k+1}, \ldots, x^n \). In particular, if \( V \) is a small neighborhood of \( o \) in \( \mathbb{R}^m \), then the mapping \( T : V \to \mathbb{R}^m \) given by
\[ T(y) := (y^1, \ldots, y^k, y^{k+1} + f^{k+1}(y^1, \ldots, y^k), \ldots, y^m + f^m(y^1, \ldots, y^k)) \]
is well defined. Now note that
\[ DT(o) = \begin{pmatrix} I_k & 0 \\ 0 & I_{m-k} \end{pmatrix}. \]

Thus, by the inverse function theorem, \( \psi := T^{-1} \) is well defined on an open neighborhood of \( o \) in \( \mathbb{R}^m \). Finally note that
\[ \psi \circ f \circ \phi^{-1}(x) = \psi(x^1, \ldots, x^k, \tilde{f}^{k+1}(x), \ldots, \tilde{f}^m(x)) = \psi \circ T(x^1, \ldots, x^k, 0, \ldots, 0) = (x^1, \ldots, x^k, 0, \ldots, 0), \]
as desired. \( \square \)
Exercise 13. Show that there exists no $C^1$ function $f: \mathbb{R}^2 \to \mathbb{R}$ which is one-to-one.