Math 550 May 7, 2002
Vector Calculus
Spring 2002, USC

Final Exam

Choose ***ONLY 10 OF THE FOLLOWING 13*** problems. In addition, you may also do problem 14.

1. Prove

(a) $\|u \times v\|^2 = \|u\|^2\|v\|^2 - (u \cdot v)^2$

(b) The diagonals of a prallelogram are orthogonal if and only if the parallelogram is a rhombus.

2. Find the distance between

(a) The point $(3, 4, 5)$ and the plane $2x + y + 3z = 5$.

(b) The lines $\ell_1(t) = t(8, -1, 0) + (-1, 3, 5)$ and $\ell_2(t) = t(0, 3, 1) + (0, 3, 4)$.

3. Evaluate

(a) $\int_{-\infty}^{\infty} e^{-x^2}\, dx$.

(b) $\int_0^2 \int_{y/2}^1 e^{-x^2}\, dx\, dy$

4. Find the center of mass of the icecream cone given by $x^2 + y^2 + z^2 \leq 1$ and $z \geq \sqrt{x^2 + y^2}$ if the density is $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

5. Find the average value of the y coordinate of the semicircle $x^2 + y^2 = 1$, $y > 0$.

6. Find the average value of the z coordinate of the hemisphere $x^2+y^2+z^2 = 1$, $z > 0$.

7. State Green's theorem and use it to compute the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

8. Show that the gravitational vector filed $\mathbf{F} := -\frac{\mathbf{r}}{\|\mathbf{r}\|^3}$ is conservative. What is the total work done in moving a particle from a point $\mathbf{r_0}$ to a point $\mathbf{r_1}$. (**Extra Credit:** Use Stokes theorem to show that if a vector field is conservative, then total work along any closed path is zero.)

9. State Gauss's theorem and use it to show that the volume of any cone with base D and height h is given by $1/3 Area(D)h$.

10. (a) Show that the length of the graph of the function $y = f(x)$, $a \leq x \leq b$, is given by $\int_a^b \sqrt{1 + f'(x)^2}\, dx$. (**Extra Credit**: Find a similar formula for the length of the graph of a curve given by the polar equation $r = f(\theta)$, $a \leq \theta \leq b$).

11. (a) Show that the area of a surface given by rotating the graph of the function $y = f(x)$, $a \leq x \leq b$, around x-axis is given by

$$2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2} dx.$$

(**Extra Credit:** Show that the area of a sphere cut by a pair of parallel planes depends only on the distance between the two planes).

12. State Stokes theorem and verify that it holds for the vector field $\mathbf{F}(x, y, z) := (x + z, y + z, z^2)$ on the hemisphere given by $x^2 + y^2 + z^2 = 1$ and $z > 1$.

13. Suppose that a particle of mass m moves on a path $\mathbf{c}(t)$ in the gravitational vectorfield $\mathbf{F}$ according to Newton's second law: $\mathbf{F}(\mathbf{c}(t)) = m\mathbf{c}''(t)$. Show that (a) the angular momentum $\mathbf{h}(t) := \mathbf{c}(t) \times \mathbf{c}'(t)$ stays constant in time, and (b) $\mathbf{c}(t) \cdot \mathbf{h}(t) = 0$. What can we conclude from (a) and (b) with regard to the path of the particle?

14. (Extra Credit). Show that there is no gravitational force inside a hollow planet.

Each problem is worth 10 points. Extra credits in problems 8, 10 and 11 is worth an additional 5 points each.