Review Problems

1. Prove
   (a) $\|u \times v\|^2 = \|u\|^2\|v\|^2 - (u \cdot v)^2$
   (b) The midpoints of a quadrilateral determine a parallelogram.
   (c) The Pythagorean theorem.
   (d) The diagonals of a parallelogram are orthogonal if and only if the parallelogram is a rhombus.

2. Find the distance between
   (a) The point $(3, 4, 5)$ and the plane $2x + y + 3z = 5$.
   (b) The lines $\ell_1(t) = t(8, -1, 0) + (-1, 3, 5)$ and $\ell_2(t) = t(0, 3, 1) + (0, 3, 4)$.
   (c) The point $(2, -1)$ and the line $\ell: x = 3t + 7, y = 5t - 3$.

3. Evaluate
   (a) $\int \int_D \sin(x^2 + y^2) \, dx \, dy$ where $D$ is the disk $x^2 + y^2 \leq \pi$.
   (b) $\int_{-\infty}^{\infty} e^{-x^2} \, dx$.
   (c) $\int_0^\pi \int_y^{\pi} \sin x \frac{dx}{x} \, dx \, dy$

4. Find the center of mass of:
   (a) The icecream cone given by $x^2 + y^2 + z^2 \leq 1$ and $z \geq \sqrt{x^2 + y^2}$ if the density is $\delta(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.
   (b) The tetrahedron with vertices $(0, 0, 0), (1, 0, 0), (0, 2, 0)$, and $(0, 0, 3)$ (just set up the integrals).

5. Find the average value of:
   (a) The $y$ coordinate of the half-disk $x^2 + y^2 \leq 1, y > 0$.
   (b) The $z$ coordinate of the half-ball $x^2 + y^2 + z^2 \leq 1, z > 0$. 
(c) The y coordinate of the semicircle $x^2 + y^2 = 1$, $y > 0$.

(d) The z coordinate of the hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$.

6. Use Greens theorem to compute the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

7. Show that the gravitational vector filed $\mathbf{F} := -\frac{r}{\|r\|^3}$ is conservative. What is the total work done in moving a particle from a point $\mathbf{r}_0$ to a point $\mathbf{r}_1$.

8. Use Gauss’s theorem to show that the volume of the cone with base $D$ and height $h$ is given by $\frac{1}{3} \text{Area}(D)h$.

9. Show that the length of the graph of the function $y = f(x)$, $a \leq x \leq b$, is given by $\int_a^b \sqrt{1 + f'(x)^2} dx$.

10. Show that the area of a surface given by rotating the graph of the function $y = f(x)$, $a \leq x \leq b$, around x-axis is given by $2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} dx$.

11. Use the previous problem to show that the area of a sphere cut by a pair of parallel planes depends only on the distance between the two planes.

12. Find $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ where $S$ is the hemisphere given by $x^2 + y^2 + z^2 = 1$ and $\mathbf{F}(x, y, z) := (x + z, y + z, z^2)$ (Hint: use Stokes theorem).

13. Suppose that a particle of mass $m$ moves on a path $\mathbf{c}(t)$ in the gravitational vectorfield $\mathbf{F}$ according to Newton’s second law: $\mathbf{F}(c(t)) = mc''(t)$. Show that (a) the angular momentum $\mathbf{h}(t) := \mathbf{c}(t) \times \mathbf{c}'(t)$ stays constant in time, and (b) $\mathbf{c}(t) \cdot \mathbf{h}(t) = 0$. What can we conclude from (a) and (b) with regard to the path of the particle?

14. Suppose that a particle of mass $m$ moves along a curve $\mathbf{c}(t)$, $a \leq t \leq b$, inside a vector field $\mathbf{F}$. Show that the total work $\int_a^b \mathbf{F} \cdot d\mathbf{s} = \frac{1}{2}mv^2(b) - \frac{1}{2}mv^2(a)$, where $v(t) = \|\mathbf{c}'(t)\|$. (Hint: Use Newton’s second law: $\mathbf{F}(c(t)) = mc''(t)$.)

15. Show that there is no gravitational force inside a hollow planet.