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Differential Geometry
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Lecture Notes 0

Basics of Euclidean Geometry

By $\mathbf{R}$ we shall always mean the set of real numbers. The set of all n-tuples of real numbers $\mathbf{R}^n := \{(p^1, \ldots, p^n) \mid p^i \in \mathbf{R}\}$ is called the *Euclidean n-space*. So we have

$$p \in \mathbf{R}^n \iff p = (p^1, \ldots, p^n), \quad p^i \in \mathbf{R}.$$

Let p and q be a pair of points (or vectors) in $\mathbf{R}^n$. We define $p + q := (p^1 + q^1, \ldots, p^n + q^n)$. Further, for any scalar $r \in \mathbf{R}$, we define $rp := (rp^1, \ldots, rp^n)$. It is easy to show that the operations of addition and scalar multiplication that we have defined turn $\mathbf{R}^n$ into a vector space over the field of real numbers. Next we define the standard *inner product* on $\mathbf{R}^n$ by

$$\langle p, q \rangle = p^1 q^1 + \ldots + p^n q^n.$$

Note that the mapping $\langle \cdot, \cdot \rangle \colon \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ is linear in each variable and is symmetric. The standard inner product induces a norm on $\mathbf{R}^n$ defined by

$$\|p\| := \langle p, p \rangle^{\frac{1}{2}}.$$

If $p \in \mathbf{R}$, we usually write $|p|$ instead of $\|p\|$.

The first nontrivial fact in Euclidean geometry, is the following important result which had numerous applications:

Theorem 1. (The Cauchy-Schwartz inequality) *For all p and q in $\mathbf{R}^n$*

$$|\langle p, q \rangle| \leqslant \|p\| \, \|q\|.$$

The equality holds if and only if $p = \lambda q$ for some $\lambda \in \mathbf{R}$.

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Since this is such a remarkable and far reaching result we will include here three different proofs. The first proof is quite short and slick, but also highly nontransparent, i.e., it is not easy to see how someone could come up with that. The second proof is perhaps more reasonable, but also more advanced. The third proof is the most elementary, but then again it is quite tricky.

Proof I (Quadratic Formula). If $p = \lambda q$ it is clear that equality holds. Otherwise, let $f(\lambda) := \langle p - \lambda q, p - \lambda q\rangle$. Then $f(\lambda) > 0$. Further, note that $f(\lambda)$ may be written as a quadratic equation in λ:

$$f(\lambda) = \|p\|^2 - 2\lambda\langle p, q\rangle + \lambda^2\|q\|^2.$$

Hence its discriminant must be negative:

$$4\langle p, q\rangle^2 - 4\|p\|^2\|q\|^2 < 0$$

which completes the proof. □

Proof II (Lagrange Multipliers). Again suppose that $p \neq \lambda q$. Then

$$\langle p, q\rangle = \|p\|\|q\| \left\langle \frac{p}{\|p\|}, \frac{q}{\|q\|} \right\rangle.$$

Thus it suffices to prove that for all unit vectors $\overline{p}$ and $\overline{q}$ we have

$$|\langle \overline{p}, \overline{q}\rangle| \leq 1,$$

and equality holds if and only if $p = \pm q$. This may be proved by using the method of lagrangne multipliers to find the maximum of the function $\langle x, y\rangle$ subject to the constraints $\|x\| = 1$ and $\|y\| = 1$. More explicitly we need to find the critical points of

$$\begin{aligned} f(x, y, \lambda_1, \lambda_2) &:= \langle x, y\rangle + \lambda_1(\|x\|^2 - 1) + \lambda_2(\|y\|^2 - 1) \\ &= \sum_{i=1}^{n}(x_i y_i + \lambda_1 x_i^2 + \lambda_2 y_i^2) - \lambda_1 - \lambda_2. \end{aligned}$$

At a critical point we must have $0 = \partial f/\partial x_i = y_i + 2\lambda_1 x_i$, which yields that $y = \pm x$. □

Proof III (Induction). First note that the case $n = 1$ is trivial. For $n = 2$, the proof amounts to showing that

$$(p_1q_1 + p_2q_2)^2 \leq (p_1^2 + p_2^2)(q_1^2 + q^2)^2.$$

This is also easily verified by the expansion and simplification of both sides which reduces the above inequality to $(p_1q_2 - q_2p_1)^2 \geq 0$. Now suppose that the inequality we like to prove holds for n. Then to prove this for $n+1$ note that

$$\begin{aligned}
\sum_{i=1}^{n+1} p_iq_i &= \sum_{i=1}^{n} p_iq_i + p_{n+1}q_{n+1} \\
&\leq \sqrt{\sum_{i=1}^{n} p_i^2}\sqrt{\sum_{i=1}^{n} q_i^2} + p_{n+1}q_{n+1} \\
&\leq \sqrt{\sum_{i=1}^{n} p_i^2 + p_{n+1}^2}\sqrt{\sum_{i=1}^{n} q_i^2 + q_{n+1}^2} \\
&= \sqrt{\sum_{i=1}^{n+1} p_i^2}\sqrt{\sum_{i=1}^{n+1} q_i^2}.
\end{aligned}$$

The first inequlity above is just the inductive step, i.e., the assumption that the inequality we want to prove holds for n, and the second inequality above is just an application of the case $n = 2$ which we established earlier. □

There is yet another proof of the Cauchy-Schwartz ineqaulity which combines ideas from the first and second proofs mentioned above, but avoids using either the quadratic formula or the Lagrange multipliers:

Exercise 2. (The simplest proof of the Cauchy-Schwartz inequality) Assume, as in the second proof above, that $\|p\| = 1 = \|q\|$ and note that

$$0 < \|p - \langle p, q\rangle q\|^2 = \big\langle p - \langle p, q\rangle q, p - \langle p, q\rangle q\big\rangle.$$

whenever $p \neq \lambda q$. Expanding the right hand side yields the desired result.

The standard Euclidean distance in $\mathbf{R}^n$ is given by

$$\mathrm{dist}(p, q) := \|p - q\|.$$

An immediate application of the Cauchy-Schwartz inequality is the following

Exercise 3. (The triangle inequality) Show that

$$\text{dist}(p, q) + \text{dist}(q, r) \geqslant \text{dist}(p, r)$$

for all p, q in $\mathbf{R}^n$.

By a *metric* on a set X we mean a mapping $d\colon X \times X \to \mathbf{R}$ such that

1. $d(p, q) \geqslant 0$, with equality if and only if $p = q$.
2. $d(p, q) = d(q, p)$.
3. $d(p, q) + d(q, r) \geqslant d(p, r)$.

These properties are called, respectively, positive-definiteness, symmetry, and the triangle inequality. The pair (X, d) is called a *metric space*. Using the above exercise, one immediately checks that $(\mathbf{R}^n, \text{dist})$ is a metric space. *Geometry*, in its broadest definition, is the study of metric spaces, and *Euclidean Geometry*, in the modern sense, is the study of the metric space $(\mathbf{R}^n, \text{dist})$.

Finally, we define the *angle* between a pair of nonzero vectors in $\mathbf{R}^n$ by

$$\text{angle}(p, q) := \cos^{-1} \frac{\langle p, q\rangle}{\|p\| \, \|q\|}.$$

Note that the above is well defined by the Cauchy-Schwartz inequality. Now we have all the necessary tools to prove the most famous result in all of mathematics:

Exercise 4. (The Pythagorean theorem) Show that in a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the length of the sides.

The next exercise is concerned with another corner stone of Euclidean Geometry; however, the proof requires the use of some trignometric identities and is computationally intensive.

Exercise* 5. (Sum of the angles in a triangle) Show that the sum of the angles in a triangle is π.