

## Lecture Notes 15

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### 2.13 The Geodesic Curvature

Let  $\alpha: I \rightarrow M$  be a unit speed curve lying on a surface  $M \subset \mathbf{R}^3$ . Then the *absolute geodesic curvature* of  $\alpha$  is defined as

$$|\kappa_g| := \|(\alpha'')^\top\| = \|\alpha'' - \langle \alpha'', n(\alpha) \rangle n(\alpha)\|,$$

where  $n$  is a local Gauss map of  $M$  in a neighborhood of  $\alpha(t)$ . In particular note that if  $M = \mathbf{R}^2$ , then  $|\kappa_g| = \kappa$ , i.e., absolute geodesic curvature of a curve on a surface is a generalization of the curvature of curves in the plane.

**Exercise 1.** Show that the absolute geodesic curvature of great circles in a sphere and helices on a cylinder are everywhere zero.

Similarly, the (*signed*) *geodesic curvature* generalizes the notion of the signed curvature of planar curves and may be defined as follows.

We say that a surface  $M \subset \mathbf{R}^3$  is *orientable* provided that there exists a (global) Gauss map  $n: M \rightarrow \mathbf{S}^2$ , i.e., a *continuous* mapping which satisfies  $n(p) \in T_p M$ , for all  $p \in M$ . Note that if  $n$  is a global Gauss map, then so is  $-n$ . In particular, any orientable surface admits precisely two choices for its global Gauss map. Once we choose a Gauss map  $n$  for an orientable surface, then  $M$  is said to be *oriented*.

If  $M$  is an oriented surface (with global Gauss map  $n$ ), then, for every  $p \in M$ , we define a mapping  $J: T_p M \rightarrow T_p M$  by

$$JV := n \times V.$$

**Exercise 2.** Show that if  $M = \mathbf{R}^2$ , and  $n = (0, 0, 1)$ , then  $J$  is clockwise rotation about the origin by  $\pi/2$ .

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Then the *geodesic curvature* of a unit speed curve  $\alpha: I \rightarrow M$  is given by

$$\kappa_g := \langle \alpha'', J\alpha' \rangle.$$

Note that, since  $J\alpha'$  is tangent to  $M$ ,

$$\langle \alpha'', J\alpha' \rangle = \langle (\alpha'')^\top, J\alpha' \rangle.$$

Further, since  $\|\alpha'\| = 1$ ,  $\alpha''$  is orthogonal to  $\alpha'$ , which in turn yields that the projection of  $\alpha''$  into the tangent plane is either parallel or antiparallel to  $J\alpha'$ . Thus  $\kappa_g > 0$  when the projection of  $\alpha''$  is parallel to  $J\alpha'$  and is negative otherwise.

Note that if the curvature of  $\alpha$  does not vanish (so that the principal normal  $N$  is well defined), then

$$\kappa_g = \kappa \langle N, JT \rangle.$$

In particular geodesic curvature is invariant under reparametrizations of  $\alpha$ .

**Exercise 3.** Let  $\mathbf{S}^2$  be oriented by its outward unit normal, i.e.,  $n(p) = p$ , and compute the geodesic curvature of the circles in  $\mathbf{S}^2$  which lie in planes  $z = h$ ,  $-1 < h < 1$ . Assume that all these circles are oriented consistently with respect to the rotation about the  $z$ -axis.

Next we derive an expression for  $\kappa_g$  which does not require that  $\alpha$  have unit speed. To this end, let  $s: I \rightarrow [0, L]$  be the arclength function of  $\alpha$ , and recall that  $\bar{\alpha} := \alpha \circ s^{-1}: [0, L] \rightarrow M$  has unit speed. Thus

$$\kappa_g = \bar{\kappa}_g(s) = \langle \bar{\alpha}''(s), J\alpha'(s) \rangle.$$

Now recall that  $(s^{-1})' = 1/\|\alpha'\|$ . Thus by chain rule.

$$\bar{\alpha}'(t) = \alpha'(s^{-1}(t)) \cdot \frac{1}{\|\alpha'(s^{-1}(t))\|}.$$

Further, differentiating both sides of the above equation yields

$$\bar{\alpha}'' = \alpha''(s^{-1}) \cdot \frac{1}{\|\alpha'(s^{-1})\|^2} + \alpha'(s^{-1}) \cdot \frac{-\langle \alpha'(s^{-1}), \alpha'(s^{-1}) \rangle}{\|\alpha'(s^{-1})\|^3}.$$

Substituting these values into the last expression for  $\bar{\kappa}_g$  above yields

$$\kappa_g = \frac{\langle \alpha'', J\alpha' \rangle}{\|\alpha'\|^3}.$$

**Exercise 4.** Verify the last two equations.

Next we show that the geodesic curvature is intrinsic, i.e., it is invariant under isometries of the surface. To this end define  $\tilde{\alpha}': \alpha(I) \rightarrow \mathbf{R}^3$  be the vectorfield along  $\alpha(I)$  given by

$$\tilde{\alpha}'(\alpha(t)) = \alpha'(t).$$

Then one may immediately check that

$$\alpha''(t) = \bar{\nabla}_{\alpha'(t)} \tilde{\alpha}'.$$

Thus

$$\langle \alpha'', J\alpha' \rangle = \langle (\alpha'')^\top, J\alpha' \rangle = \langle \nabla_{\alpha'} \tilde{\alpha}', J\alpha' \rangle.$$

and it follows that

$$\kappa_g = \frac{\langle \nabla_{\alpha'} \tilde{\alpha}', J\alpha' \rangle}{\|\alpha'\|^3}.$$

We say that a curve  $\alpha: I \rightarrow M$  is a geodesic provided that its geodesic curvature  $\kappa_g \equiv 0$ .

**Exercise 5.** Show that if  $\alpha$  is a geodesic, then it must have constant speed.

**Exercise 6.** Show that if  $\alpha$  is parametrized by arclength, then

$$|\kappa_g| = \|\nabla_{\alpha'} \tilde{\alpha}'\|.$$

**Exercise 7.** Show that  $\alpha$  is a geodesic if and only if  $\nabla_{\alpha'} \tilde{\alpha}' \equiv 0$ .

Now recall that  $\nabla$  is intrinsic, which immediately implies that  $|\kappa_g|$  is intrinsic by the last exercise. Thus to complete the proof that  $\kappa_g$  is intrinsic it remains to show that  $J$  is intrinsic. To see this let  $X: U \rightarrow M$  be a local patch, then

$$JX_i = \sum_{j=1}^2 b_{ij} X_j.$$

In particular,

$$JX_1 = b_{11}X_1 + b_{12}X_2.$$

Now note that

$$0 = \langle JX_1, X_1 \rangle = b_{11}g_{11} + b_{12}g_{21}.$$

Further,

$$g_{11} = \langle X_1, X_1 \rangle = \langle JX_1, JX_1 \rangle = b_{11}^2 g_{11} + 2b_{11}b_{12}g_{12} + b_{12}^2 g_{22}.$$

Solving for  $b_{21}$  in the next to last equation, and substituting in the last equation yields

$$g_{11} = b_{11}^2 g_{11} - 2b_{11}^2 g_{11} + b_{11}^2 \frac{g_{11}^2}{g_{21}^2} g_{22} = b_{11}^2 \left( -g_{11} + \frac{g_{11}^2}{g_{21}^2} g_{22} \right).$$

Thus  $b_{11}$  may be computed in terms of  $g_{ij}$  which in turn yields that  $b_{12}$  may be computed in terms of  $g_{ij}$  as well. So  $JX_1$  may be expressed intrinsically. Similarly,  $JX_2$  may be expressed intrinsically as well. So we conclude that  $J$  is intrinsic.

## 2.14 Geodesics in Local Coordinates

Here we will derive a system of ordinary differential equations, in terms of any local coordinates, whose solutions yield geodesics.

To this end let  $X: U \rightarrow M$  be a patch, and  $\alpha: I \rightarrow X(U)$  be a unit speed one-to-one curve. Then we may write

$$X(u(t)) = \alpha(t),$$

by letting  $u(t) := X^{-1}(\alpha(t))$ . Next note that, if  $u_i$  denotes the coordinates of  $u$ , i.e.,  $u(t) = (u_1(t), u_2(t))$ , then by the chain rule,

$$\alpha' = \sum_{i=1}^2 X_i u'_i,$$

which in turn yields

$$\alpha'' = \sum_{i,j=1}^2 X_{ij} u'_i u'_j + X_i u''_i = \sum_{i,j,k=1}^2 (\Gamma_{ij}^k X_k + \ell_{ij} N) u'_i u'_j + X_i u''_i,$$

by the Gauss's formula. Consequently,

$$(\alpha'')^\top = \sum_{i,j,k=1}^2 (\Gamma_{ij}^k X_k) u'_i u'_j + X_i u''_i = \sum_{i,j,k=1}^2 (\Gamma_{ij}^k u'_i u'_j + u''_i) X_k.$$

So, since  $|\kappa_g| = \|(\alpha'')^\top\|$ , we conclude that  $\alpha$  is a geodesic if and only if

$$\sum_{i,j=1}^2 (\Gamma_{ij}^k u'_i u'_j + u''_k) = 0$$

for  $k = 1, 2$ . In other words, for  $\alpha$  to be a geodesic the following two equations must be satisfied:

$$\begin{aligned} u''_1 + \Gamma_{11}^1 (u'_1)^2 + 2\Gamma_{12}^1 u'_1 u'_2 + \Gamma_{22}^1 (u'_2)^2 &= 0 \\ u''_2 + \Gamma_{11}^2 (u'_1)^2 + 2\Gamma_{12}^2 u'_1 u'_2 + \Gamma_{22}^2 (u'_2)^2 &= 0 \end{aligned}$$

**Exercise 8.** Write down the equations of the geodesic in a surface of revolution. In particular, verify that the great circles in a sphere are geodesics.

## 2.15 Parallel Translation

Here we will give another interpretation for the concept of geodesic curvature. Let  $\alpha: I \rightarrow M$  be a simple curve and  $V$  be a vector field on  $M$ . We say that  $V$  is *parallel* along  $\alpha$  provided that

$$\nabla_{\alpha'(t)} V = ((V \circ \alpha)')^\top = 0.$$

for all  $t \in I$ . Recall that  $\alpha$  is a geodesic if and only if its velocity is parallel (i.e.,  $\nabla_{\alpha'(t)} \alpha' \equiv 0$ )

**Exercise 9.** Show that if  $V$  is parallel along  $\alpha$ , then its length is constant.

**Exercise 10.** Show that if  $V$  and  $W$  are a pair of parallel vector fields along  $\alpha$ , then the angle between them is constant.

**Proposition 11.** *If  $\alpha$  is a unit speed curve on a surface, and  $V$  is a parallel vector field along  $\alpha$ , which makes an angle  $\phi$  with the tangent vector of  $\alpha$ , then  $\kappa_g = \phi'$ .*

*Proof.* We may assume that  $V$  has unit length. Then we may write:

$$V = \cos(\phi)T + \sin(\phi)JT.$$

Computing  $\|\nabla_T V\|$ , and setting it equal to zero yields the desired result.  $\square$

**Corollary 12.** *The total geodesic curvature of a curve on a surface is equal to the total rotation of a parallel vector field along the curve, i.e.*

$$\int_a^b \kappa_g = \phi(b) - \phi(a).$$

where  $\phi$  is an angle function between  $T$  and  $V$ .