

# Lecture Notes 7

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## 1.17 The Frenet-Serret Frame and Torsion

Recall that if  $\alpha: I \rightarrow \mathbf{R}^n$  is a unit speed curve, then the unit tangent vector is defined as

$$T(t) := \alpha'(t).$$

Further, if  $\kappa(t) = \|T'(t)\| \neq 0$ , we may define the principal normal as

$$N(t) := \frac{T'(t)}{\kappa(t)}.$$

As we saw earlier, in  $\mathbf{R}^2$ ,  $\{T, N\}$  form a moving frame whose derivatives may be expressed in terms of  $\{T, N\}$  itself. In  $\mathbf{R}^3$ , however, we need a third vector to form a frame. This is achieved by defining the *binormal* as

$$B(t) := T(t) \times N(t).$$

Then similar to the computations we did in finding the derivatives of  $\{T, N\}$ , it is easily shown that

$$\begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix}' = \begin{pmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{pmatrix} \begin{pmatrix} T(t) \\ N(t) \\ B(t) \end{pmatrix},$$

where  $\tau$  is the *torsion* which is defined as

$$\tau(t) := -\langle B', N \rangle.$$

Note that torsion is well defined only when  $\kappa \neq 0$ , so that  $N$  is defined. Torsion is a measure of how much a space curve deviates from lying in a plane:

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**Exercise 1.** Show that if the torsion of a curve  $\alpha: I \rightarrow \mathbf{R}^3$  is zero everywhere then it lies in a plane. (*Hint:* We need to check that there exist a point  $p$  and a (fixed) vector  $v$  in  $\mathbf{R}^3$  such that  $\langle \alpha(t) - p, v \rangle = 0$ . Let  $v = B$ , and  $p$  be any point of the curve.)

**Exercise 2.** Computer the curvature and torsion of the circular helix

$$(r \cos t, r \sin t, ht)$$

where  $r$  and  $h$  are constants. How does changing the values of  $r$  and  $h$  effect the curvature and torsion.

## 1.18 Curves of Constant Curvature and Torsion

The above exercise shows that the curvature and torsion of a circular helix are constant. The converse is also true

**Theorem 3.** *The only curve  $\alpha: I \rightarrow \mathbf{R}^3$  whose curvature and torsion are nonzero constants is the circular helix.*

The rest of this section develops a number of exercises which leads to the proof of the above theorem

**Exercise 4.** Show that  $\alpha: I \rightarrow \mathbf{R}^3$  is a circular helix (up to rigid motion) provided that there exists a vector  $v$  in  $\mathbf{R}^3$  such that

$$\langle T, v \rangle = \text{const},$$

and the projection of  $\alpha$  into a plane orthogonal to  $v$  is a circle.

So first we need to show that when  $\kappa$  and  $\tau$  are constants,  $v$  of the above exercise can be found. We do this with the aid of the Frenet-Serret frame. Since  $\{T, N, B\}$  is an orthonormal frame, we may write

$$v = a(t)T(t) + b(t)N(t) + c(t)B(t).$$

Next we need to find  $a$ ,  $b$  and  $c$  subject to the conditions that  $v$  is a constant vector, i.e.,  $v' = 0$ , and that  $\langle T, v \rangle = \text{const}$ . The latter implies that

$$a = \text{const}$$

because  $\langle T, v \rangle = a$ . In particular, we may set  $a = 1$ .

**Exercise 5.** By setting  $v' = 0$  show that

$$v = T + \frac{\kappa}{\tau}B,$$

and check that  $v$  is the desired vector, i.e.  $\langle T, v \rangle = \text{const}$  and  $v' = 0$ .

So to complete the proof of the theorem, only the following remains:

**Exercise 6.** Show that the projection of  $\alpha$  into a plane orthogonal to  $v$ , i.e.,

$$\bar{\alpha}(t) := \alpha(t) - \langle \alpha(t), v \rangle \frac{v}{\|v\|^2}$$

is a circle. (*Hint:* Compute the curvature of  $\bar{\alpha}$ .)

## 1.19 Intrinsic Characterization of Spherical Curves

In this section we derive a characterization in terms of curvature and torsion for unit speed curves which lie on a sphere. Suppose  $\alpha: I \rightarrow \mathbf{R}^3$  lies on a sphere of radius  $r$ . Then there exists a point  $p$  in  $\mathbf{R}^3$  (the center of the sphere) such that

$$\|\alpha(t) - p\| = r.$$

Thus differentiation yields

$$\langle T(t), \alpha(t) - p \rangle = 0.$$

Differentiating again we obtain:

$$\langle T'(t), \alpha(t) - p \rangle + 1 = 0.$$

The above expression shows that  $\kappa(t) \neq 0$ . Consequently  $N$  is well defined, and we may rewrite the above expression as

$$\kappa(t)\langle N(t), \alpha(t) - p \rangle + 1 = 0.$$

Differentiating for the third time yields

$$\kappa'(t)\langle N(t), \alpha(t) - p \rangle + \kappa(t)\langle -\kappa(t)T(t) + \tau(t)B(t), \alpha(t) - p \rangle = 0,$$

which using the previous expressions above we may rewrite as

$$-\frac{\kappa'(t)}{\kappa(t)} + \kappa(t)\tau(t)\langle B(t), \alpha(t) - p \rangle = 0.$$

Next note that, since  $\{T, N, B\}$  is orthonormal,

$$\begin{aligned} r^2 &= \|\alpha(t) - p\|^2 \\ &= \langle \alpha(t) - p, T(t) \rangle^2 + \langle \alpha(t) - p, N(t) \rangle^2 + \langle \alpha(t) - p, B(t) \rangle^2 \\ &= 0 + \frac{1}{\kappa^2(t)} + \langle \alpha(t) - p, B(t) \rangle^2. \end{aligned}$$

Thus, combining the previous two calculations, we obtain:

$$-\frac{\kappa'(t)}{\kappa(t)} + \kappa(t)\tau(t)\sqrt{r^2 - \frac{1}{\kappa^2(t)}} = 0.$$

**Exercise 7.** Check the converse, that is supposing that the curvature and torsion of some curve satisfies the above expression, verify whether the curve has to lie on a sphere of radius  $r$ .

To do the above exercise, we need to first find out where the center  $p$  of the sphere could lie. To this end we start by writing

$$p = \alpha(t) + a(t)T(t) + b(t)N(t) + c(t)B(t),$$

and try to find  $a(t)$ ,  $b(t)$  and  $c(t)$  so that  $p' = (0, 0, 0)$ , and  $\|\alpha(t) - p\| = r$ . To make things easier, we may note that  $\alpha(t) = 0$  (why?). Then we just need to find  $b(t)$  and  $c(t)$  subject to the two constraints mentioned above. We need to verify whether this is possible, when  $\kappa$  and  $\tau$  satisfy the above expression.

## 1.20 The Local Canonical form

In this section we show that all  $C^3$  curve in  $\mathbf{R}^3$  essentially look the same in the neighborhood of any point which has nonvanishing curvature and a given sign for torsion.

Let  $\alpha: (-\epsilon, \epsilon) \rightarrow \mathbf{R}^3$  be a  $C^3$  curve. By Taylor's theorem

$$\alpha(t) = \alpha(0) + \alpha'(0)t + \frac{1}{2}\alpha''(0)t^2 + \frac{1}{6}\alpha'''(0)t^3 + R(t)$$

where  $\lim_{t \rightarrow 0} |R(t)|/t^3 = 0$ , i.e., for  $t$  small, the remainder term  $R(t)$  is negligible. Now suppose that  $\alpha$  has unit speed. Then

$$\begin{aligned}\alpha' &= T \\ \alpha'' &= T' = \kappa N \\ \alpha''' &= (\kappa N)' = \kappa' N + \kappa(-\kappa T + \tau B) = -\kappa^2 T + \kappa' N + \tau B.\end{aligned}$$

So we have

$$\begin{aligned}\alpha(t) &= \alpha(0) + T_0 t + \frac{\kappa_0 N_0 t^2}{2} + \frac{(-\kappa_0^2 T_0 + \kappa_0' N_0 + \tau_0 B_0) t^3}{6} + R(t) \\ &= \alpha(0) + \left(t - \frac{\kappa_0^2}{6} t^3\right) T_0 + \left(\frac{\kappa_0}{2} t^2 + \frac{\kappa_0'}{6} t^3\right) N_0 + \left(\frac{-\kappa_0 \tau_0}{6} t^3\right) B_0 + R(t)\end{aligned}$$

Now if, after a rigid motion, we suppose that  $\alpha(0) = (0, 0, 0)$ ,  $T = (1, 0, 0)$ ,  $N = (0, 1, 0)$ , and  $B = (0, 0, 1)$ , then we have

$$\alpha(t) = \left( t - \frac{\kappa_0^2}{6} t^3 + R_x, \frac{\kappa_0}{2} t^2 + \frac{\kappa_0'}{6} t^3 + R_y, \frac{-\kappa_0 \tau_0}{6} t^3 + R_z \right),$$

where  $(R_x, R_y, R_z) = R$ . It follows then that when  $t$  is small

$$\alpha(t) \approx \left( t, \frac{\kappa_0}{2} t^2, \frac{-\kappa_0 \tau_0}{6} t^3 \right).$$

Thus, up to third order of differentiation, any curve with nonvanishing curvature in space may be approximated by a cubic curve. Also note that the above approximation shows that any planar curve with nonvanishing curvature locally looks like a parabola.

**Exercise 8.** Show that the curvature of a space curve  $\alpha$  at a point  $t_0$  with nonvanishing curvature is the same as the curvature of the projection of  $\alpha$  into the the osculating plane at time  $t_0$ . (The osculating plane is the plane generated by  $T$  and  $N$ .)