OPEN PROBLEMS IN
GEOMETRY OF CURVES AND SURFACES

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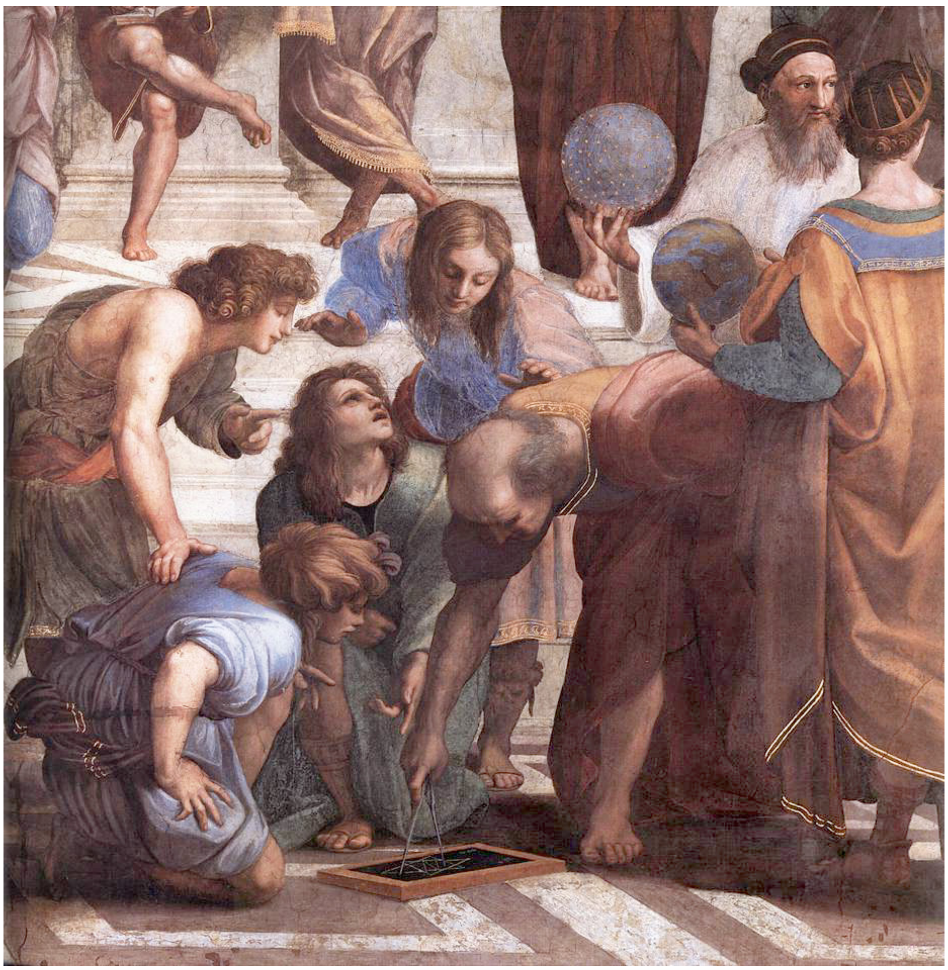

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There are problems to whose solution I would attach an infinitely greater importance than to those of mathematics, for example touching ethics, or our relation to God, or concerning our destiny and our future; but their solution lies wholly beyond us and completely outside the province of science.

–Karl Friedrich Gauss (1777-1855), Quoted in J.R. Newman, *The World of Mathematics*

It is difficult and often impossible to judge the value of a problem correctly in advance; for the final award depends upon the gain which science obtains from the problem. Nevertheless we can ask whether there are general criteria which mark a good mathematical problem. An old French mathematician said: "A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street."This clearness and ease of comprehension, here insisted on for a mathematical theory, I should still more demand for a mathematical problem if it is to be perfect; for what is clear and easily comprehended attracts, the complicated repels us.

–David Hilbert, Address to the International Congress of Mathematicians, 1900

Our Euclidean intuition, probably, inherited from ancient primates, might have grown out of the first seeds of space in the motor control systems of early animals who were brought up to sea and then to land by the Cambrian explosion half a billion years ago. Primates brain had been lingering for 30-40 million years. Suddenly, in a flash of one million years, it exploded into growth under relentless pressure of the sexual-social competition and sprouted a massive neocortex (70% neurons in humans) with an inexplicable capability for language, sequential reasoning and generation of mathematical ideas. Then Man came and laid down the space on papyrus in a string of axioms, lemmas and theorems around 300 B.C. in Alexandria.

–Misha Gromov, *Spaces and Questions*, 1999

The life so short, the craft so long to learn.

–Chaucer, *Parlement of Foules*, 1382

About the cover: Detail from *The School of Athens* by Raphael, 1511

Abstract. We collect dozens of well-known and not so well-known fundamental unsolved problems involving low dimensional submanifolds of Euclidean space. The list includes selections from differential geometry, Riemannian geometry, metric geometry, discrete or polyhedral geometry, geometric knot theory, theory of convex bodies, and integral geometry. The common thread through these selections are the simplicity and intuitive nature of the questions. Extensive bibliography and historical background are included for each set of problems.

Contents

0. Introduction

Here we discuss a number of unsolved problems in geometry of curves and surfaces which have been of interest to the author over the years. Needless to say, this selection reflects the author's personal taste and (limited) perspective, although an effort has been made to include some of the oldest and best known problems in the field. Virtually all of these questions are concerned with objects in the Euclidean plane or 3-dimensional space, although in some cases higher dimensional analogues are discussed as well, and in some cases the problems are essentially intrinsic or independent of the ambient space.

0.1. **Why study curves and surfaces?** Curves and surfaces are to geometry what numbers are to algebra. They form the basic ingredients of our visual perception and inspire the development of far reaching mathematical tools. Yet despite centuries of pure study, not to mention a wealth of growing applications in science and technology, there are still numerous open problems in this area which are strikingly intuitive and elementary to state, pointing to fundamental gaps in our conceptions of space and shape. According to Ezra Pound, "music begins to atrophy when it departs too far from the dance" and "poetry begins to atrophy when it gets too far from music". Perhaps the same can be said of geometry, and indeed all of mathematics, if it looses sight of its natural building blocks and connections to the physical world.

0.2. **Other sources for problems.** There are many lists of problems in geometry and closely related fields. For a wide range of questions in differential, Riemannian and metric geometry see Yau [166, 190], Donaldson [48], and Gromov [82, 83, 84]. Some classical problems in differential geometry and many references may be found in books by Berger [21], and Burago and Zalgaller [191]. For problems involving geodesics see Burns and Matveev [28]. A large collection of problems in discrete and convex geometry are contained in the book of Croft, Falconer, and Guy [42]; also see Gardner [58] for problems involving convex bodies. For problems in minimal surface theory see the lists by Meeks [125, 126, 129], and for surfaces of constant mean curvature see Lopez [117]. A vast collection of problems in low dimensional topology is maintained by Rob Kirby [107]. For some problems in geometric knot theory see Adams [4].

0.3. **Useful websites.** There are a number of websites for tracking geometric problems. The Open Problems Project [45], maintained by Demaine, Mitchell, O'Rourke, contains a wealth of problems in computational geometry. There are also growing lists of geometric problems on Wikipedia's Unsolved Problems [1] page. Some open problem in low dimensional topology are maintained at the Low Dimensional Topology [3] page. Finally numerous problems in all aspects of geometry are continually discussed on MathOverflow [2].

0.4. **The earlier draft of this article.** This article is a major expansion and revision of an earlier list of problems [64] which the author had collected in the Fall of 2004 for his students in a class on differential geometry in the MASS program at Penn State University. Since then several of the problems in that list have been solved: John Pardon [146] solved Gromov's question on the distortions of knots [64, Prob. 5]; Wegner [186] found examples of bicycle curves with rotation number, or relative density, different from 1/2 [64, Prob. 1], Wilmore's conjecture [64, Prob. 18] was solved by Coda and Neves [122], and the converse to the four vertex theorem for simple closed curves [64, Prob. 18] was proved by Dahlberg [43, 47].

1. Isometric Embeddings

A fundamental theme in surface theory, which dates back to the development of this subject by Gauss [59], is determining the extent to which the intrinsic geometry or metric structure of a surface contributes to its global shape in the ambient space. Questions of this type, which we discuss below may be phrased within the context of the uniqueness and existence of isometric embeddings of 2-dimensional Riemannian manifolds in Euclidean 3-space $\mathbf{R}^3$.

1.1. **Flexibility of closed surfaces.** As every child soon learns, an egg shell is not flexible. The mathematical reason is the rigidity theorem for convex surfaces first proved by Cauchy for polyhedra, then extended to smooth surfaces by Hilbert, Cohn-Vossen, Weyl, Nirenberg, and finally generalized to all convex surfaces by Alexandrov and Pogorelov [94, 9]. This theorem states that isometric (closed) convex surfaces are congruent, or, to put it more succinctly, convex surfaces are (isometrically) *rigid*. Finding an analogue of this result for general (nonconvex) surfaces is one of the oldest problems in geometry [190], [188, Problem 50], which may be traced back to Euler [54, p. 494–496] for polyhedral surfaces, see [76, 78, 115], and Maxwell [124] for smooth surfaces; however, to quote Chern [32, p. 211], "practically nothing is known" about it.

To state the problem explicitly, let us say that a surface is (isometrically) *flexible*, within a given smoothness class, if it admits a continuous deformation which preserves its smoothness and does not change its intrinsic metric. For instance, a sheet of paper is flexible in the class of $\mathcal{C}^\infty$ surfaces, since it can be rolled into a cylinder. It is also well-known, though not all obvious, that a spherical cap (a piece of sphere cut off by a plane) is smoothly flexible. No one, however, has ever found a flexible *closed* surface in the smoothness class $\mathcal{C}^2$, or even $\mathcal{C}^{1,1}$, where by closed we mean compact without boundary. Hence the following question:

Problem 1.1 (Euler [54], 1776; and Maxwell [124], 1819)**.** *Does there exist a closed $\mathcal{C}^2$ surface in Euclidean space $\mathbf{R}^3$ which is flexible?*

Note that flexibility is a stronger notion than nonrigidity. Indeed it is easy to construct $\mathcal{C}^\infty$ surfaces which are not rigid: consider for instance a closed surface with a dimple which is surrounded by a flat rim, and replace the dimple by its reflection. Such a transformation, however, cannot be carried out continuously for it would force the mean curvature vector to switch sides. One can even construct closed *analytic* surfaces which are not rigid [151], though these constructions are far more subtle.

Secondly, it is important to note that the answer to the analogue of Problem 1.1 in the category of polyhedral surfaces is yes! The first example of a flexible closed polyhedral surface was constructed by Robert Connelly [39]; however, these flexible polyhedra are not so natural. Indeed Gluck [76] had shown that almost all closed polyhedral surfaces are rigid, see the book of Igor Pak [145]. Further, Stoker [178] explicitly described a large class of nonconvex polyhedral surfaces which are rigid, which were further studied and generalized by Rodriguez and Rosenberg [153]. Stoker polyhedra have vertices which are either convex or are of saddle type. The latter means that the vertex has degree 4 and the edges around that vertex alternate between valley and ridge types. Saddle vertices have the property that unit normals to the faces around the vertex form the vertices of a convex geodesic polygon in the sphere. This local convexity property of the Gauss map is the key ingredient to generalizing Cauchy's rigidity theorem to nonconvex polyhedra [153]. Rigidity of some other classes of nonconvex polyhedra have also been established in papers of Schlenker with Ismetsiev [103] and Connolly [40].

An even greater surprise, than flexibility of nonconvex polyhedra, is the flexibility of $\mathcal{C}^1$ isometric embeddings. For instance one can squeeze the unit sphere into an an arbitrarily small ball through a continuous one-parameter family of isometric $\mathcal{C}^1$ immersions [23]! Thus the $\mathcal{C}^2$ requirement in the above problem is not superfluous. The remarkable nature of $\mathcal{C}^1$ isometric embeddings were first discovered by Nash and Kuiper, and further developed by Gromov [81] within the frame work of the h-Principle theory; see also the book by Eliashberg and Mishachev [53]. In this context, the flexibility of the $\mathcal{C}^1$ isometric embeddings corresponds to the existence of a parametric h-principle for these maps.

The fourth important point to keep in mind with regard to Problem 1.1, is that it is important that the ambient space be $\mathbf{R}^3$. Indeed Pinkall [147] showed that flat tori flex isometrically in $\mathbf{S}^3$—a phenomenon which was studied further by Kitagawa [108].

Finally we should mention the paper by Almgren and Rivin where it is shown that the mean curvature integral must be preserved under isometric flexing [12].

1.2. **Rigidity of tight surfaces.** Traditional approaches for studying Problem 1.1 consist of breaking the surface up into negatively curved and positively curved regions, and studying the flexibility of these regions individually. This method was used by Alexandrov [7] to establish the rigidity of analytic surfaces with total positive curvature 2π (e.g., tori of revolution) which are also known as *tight surfaces* [33, 113, 29], and are natural higher genus analogues of convex surfaces. Subsequently, Nirenberg [140, 94] made a beautiful but unsuccessful attempt at generalizing Alexandrov's result to smooth surfaces. Thus we have

Problem 1.2 (Alexandrov [7] 1938; and Nirenberg [140] 1962)**.** *Are all smooth tight surfaces in* $\mathbf{R}^3$ *rigid?*

In the polyhedral case Banchoff has shown that the answer to the above question is no [17]. Nirenberg observed that the convex part of smooth tight surfaces are rigid, thus the above problem comes down to the following question:

Problem 1.3. *Are negatively curved annuli bounded by a pair of fixed convex planar curves rigid?*

The main obstacle Nirenberg encountered in answering this question could be lifted by a negative answer to the following question: *Can a negatively curved annulus in* $\mathbf{R}^3$*, bounded by a pair of convex planar curves, have a closed asymptotic curve?* A curve in a surface is *asymptotic* if the normal curvature of the surface always vanishes in directions tangent to the curve. One way to approach the above problem is as follows. Suppose that we have a smooth closed curve Γ immersed in $\mathbf{R}^3$ which admits a continuous family of osculating planes, i.e., planes which contain the first and second derivatives of Γ. Then the binormal vector field of Γ may be defined as any continuous vector field which is orthogonal to these osculating planes. If Γ is an asymptotic curve in a surface M, then the gauss map of M gives a binormal vector field along Γ. Thus Nirenberg's problem leads to the following question:

Problem 1.4. *Let* Γ *be a smooth closed curve immersed in* $\mathbf{R}^3$*. Suppose that* Γ *has a continuous binormal vector field* B *which is one-to-one. Does it follow then that the ribbon* (Γ, B) *is twisted? In other words, must the linking number between* Γ *and* $\Gamma + \epsilon B$ *be nonzero?*

A positive answer to the last question will settle Problem 1.2. This approach has been discussed in [112]; however, subject to regularity assumptions not warranted by the question. Further, existence of closed asymptotic curves on negatively curved surfaces has also been studied by Arnold [16] who raised a number of interesting questions as well.

1.3. **Surfaces with prescribed boundary.** On the other hand, understanding the flexibility of positively curved surfaces with boundary is also important as far as Problems 1.1 and 1.2 are concerned. Here, a fundamental questions is:

Problem 1.5 (Yau [189], 1990)**.** *Given a metric of positive curvature on the disk what is the condition on a space curve to form the boundary of an isometric embedding of the disk?*

A nontrivial *necessary* condition has been discovered by H. Rosenberg [157], involving the self-linking number, and a result of the author [61] provides a *sufficient* criterion; however, a complete characterization is not yet known. In particular, Gluck and Pan [77] constructed an example to show that Rosenberg's condition was not sufficient. Also we should note that Guan and Spruck [87], as well as Trudinger and Wang [184], have shown that if a curve bounds a surface of positive curvature, then it bounds a surface of *constant* positive curvature. Thus Problem 1.5 may be regarded as a boundary value problem for a PDE of Monge-Ampere type. Another question related to Problem 1.5 is:

Problem 1.6 (Rosenberg [157], 1993)**.** *Does every curve bounding a surface of positive curvature in 3-space have (at least) four points where the torsion vanishes?*

The author has shown recently that the answer to the last question is yes when the surface is topologically a disk [72], through a comprehensive study of the structure of convex caps in locally convex surfaces. Convex caps play a major role in the seminal works of Alexandrov [10] and Pogorelov [149] on the isometric embeddings of convex surfaces, as well as in other fundamental results in this area such as the works by van Heijenoort [185], Sacksteder [160], and Volkov [9, Sec. 12.1]. In these studies, however, the underlying Riemannian manifolds are assumed to be complete, or nearly complete, as in the works of Greene and Wu [79, 80]. In [72], on the other hand, the author studies caps of manifolds with boundary, as in the author's previous work with Alexander [5] and Alexander and Wong [6].

The author's solution to the last problem in the genus zero case establishes a far reaching generalization of the classical four vertex theorem for planar curves. The study of special points of curvature and torsion of closed curves has generated a vast and multifaceted literature since the works of Mukhopadhyaya [135] and A. Kneser [111] on vertices of planar curves were published in 1910–1912, although aspects of these investigations may be traced even further back to the study of inflections by Möbius [133] and Klein [109], see [69, 68]. The first version of the four vertex theorem for space curves, which was concerned with curves lying on smooth strictly convex surfaces, was stated by Mohrmann [134] in 1917, and proved by Barner and Flohr [19] in 1958. This result was finally extended to curves lying on the boundary of any convex body by Sedykh [169] in 1994, after partial results by other authors [141, 22], see also Romero-Fuster and Sedykh [154] for further refinements.

Among various applications of four vertex theorems, we mention a paper of Berger and Calabi et al. [150] on physics of floating bodies, and recent work of Bray and Jauregui [26] in general relativity. See also the works of Arnold [14, 15] for relations with contact geometry, the book of Ovsienko and Tabachnikov [144] for projective geometric aspects, Angenent [13] for connections with mean curvature flow, which are also discussed in [68], and Ivanisvili et al. [100, 101] for applications to the study of Bellman functions. Other references and more background on four vertex theorems may be found in [47, 181, 67, 145, 75].

1.4. **Rigidity of Punctured surfaces.** Greene and Wu showed that smooth convex surfaces with finitely many points deleted remain rigid [79, 80]. So it is natural to wonder:

Problem 1.7. *Are there some nonconvex surfaces which remain rigid after finitely many points of them have been deleted. For instance, are punctured analytic tight surfaces, such as a torus of revolution, rigid?*

1.5. **Existence problems.** By a celebrated theorem of Nash every Riemannian manifold admits an isometric embedding into a Euclidean space of sufficiently high dimension; however, much remains unknown about the smallest dimension where this is possible for a given class of manifolds [81, 84], including 2-dimensional manifolds or surfaces.

Problem 1.8 (The global isometric embedding problem, Yau [189] 1993; Gromov [82])**.** *Can every $\mathcal{C}^\infty$ 2-dimensional Riemannian manifold be isometrically embedded in $\mathbf{R}^4$?*

It is well-known that locally the answer to the above question is yes; however, the main local problem is the following:

Problem 1.9 (The local isometric embedding problem)**.** *Given a $\mathcal{C}^\infty$ metric in a neighborhood of a point in a 2-dimensional Riemannian manifold, does there exist an isometric embedding of some neighborhood of that point into $\mathbf{R}^3$?*

The last problem has been well studied, see the book by Han and Hong for references [94]. In particular, it is well-known that the answer to the last problem is yes if the curvature is strictly positive, or strictly negative. Further, in the cases where the curvature "changes sign cleanly", i.e., 0 is a regular value of the curvature function, it is also known that the answer is yes. On the other hand, Pogorelov [148] constructed a $\mathcal{C}^2$ metric which does not admit a local isometric embedding in $\mathbf{R}^3$. See also the paper by Nadirashvilli and Yuan [137] for more results in this direction.

2. Spherical Images of Surfaces

Most problems in surface theory, including the rigidity problems mentioned earlier, may be rephrased or studied in terms of the unit normal vector field or the Gauss map of surfaces. Here we discuss three problems in this area, which may also be classified as questions in convex integration theory.

2.1. **Directed immersions.** To every ($\mathcal{C}^1$) immersion $f\colon M^n \to \mathbf{R}^{n+1}$ of a closed oriented n-manifold M, there corresponds a unit normal vector field or *Gauss map* $G_f\colon M \to \mathbf{S}^n$, which generates a set $G_f(M) \subset \mathbf{S}^n$ known as the *spherical image* of f. Conversely, one may ask:

Problem 2.1 (Gromov [82])**.** *For which sets $A \subset \mathbf{S}^n$ is there an immersion $f\colon M \to \mathbf{R}^{n+1}$ such that $G_f(M) \subset A$?*

Such a mapping would be called an *A-directed immersion* of M [53, 81, 158, 177]. It is well-known that when $A \neq \mathbf{S}^n$, f must have double points, and M must be parallelizable, e.g., M can only be the torus $\mathbf{T}^2$ when $n = 2$. Furthermore, the only known necessary condition on A is the elementary observation that $A \cup -A = \mathbf{S}^2$, while there is also a sufficient condition due to Gromov [81, Thm. (D'), p. 186]: *$A \subset \mathbf{S}^n$ is open, and there is a point $p \in \mathbf{S}^n$ such that the intersection of A with each great circle passing through p includes a (closed) semicircle.* Note that, when $n \geq 2$, examples of sets $A \subset \mathbf{S}^n$ satisfying this condition include those which are the complement of a finite set of points without antipodal pairs. Thus the spherical image of a closed hypersurface can be remarkably flexible. Like most h-principle or convex integration type arguments, however, the proof does not yield specific examples. It is therefore natural to ask [81]: "Is there a 'simple' immersion $\mathbf{T}^2 \to \mathbf{R}^3$ whose spherical image misses the four vertices of a regular tetrahedron in $\mathbf{S}^2$?" The author has given an affirmative answer to this question [66], and more generally has given a short constructive proof of the sufficiency of a slightly stronger version of Gromov's condition mentioned above for the existence of A-directed immersions of parallelizable manifolds $M^{n-1} \times \mathbf{S}^1$, where M^{n-1} is closed and orientable.

2.2. **The shadow problem.** Next we describe a problem which involves the notion of *shadows* or *shades* on illuminated surfaces and has applications to the so called "shape from shading" problems in computer vision [98], and various problems in calculus variations [34, 60]. Let $f\colon M \to \mathbf{R}^{n+1}$ be a closed oriented n-dimensional hypersurface in Euclidean space as before and $G_f\colon M \to \mathbf{S}^n$ be its Gauss map. Then for every unit vector $u \in \mathbf{S}^n$ (corresponding to the direction of light) the *shadow* (or *shade* as it is known in computer vision [98]) is defined as

$$S_u := \{\, p \in M \mid \langle G_f(p), u\rangle > 0 \,\}, \tag{1}$$

where $\langle\cdot,\cdot\rangle$ is the standard innerproduct. In 1978, motivated by problems concerning the stability of constant mean curvature (CMC) surfaces (or soap films) [187], H. Wente formulated the following question, which has since become known as the "shadow problem:

Problem 2.2 (Wente [187], 1978)**.** *Does connectedness of the shadows imply that $f(M)$ is convex?*

In [63] the author found a complete solution to this problem in 3-space: In $\mathbf{R}^3$, the answer to Problem 2.2 is yes provided that each shadow is *simply* connected, or that M is topologically a sphere; otherwise, the answer is no. In particular, there is a smooth embedded topological torus in $\mathbf{R}^3$ which has connected shadows in every direction. Indeed as the author later showed [65], every closed orientable 2-manifold admits a smooth embedding in $\mathbf{R}^3$ which has connected shadows in every direction, thus disproving a conjecture of J Choe [34]. But there is nothing known in higher dimensions.

2.3. **Minkowski problem for nonconvex surfaces.** The famous problem of Minkowski, which has been completely solved [165], is concerned with proving that every convex surface is uniquely determined by its curvature prescribed on the sphere via the Gauss map. The discrete version of this problem states that two convex polytopes whose faces are parallel and have the same area are congruent. As has been pointed out by Yau [188] it would be interesting to find analogues of these results for nonconvex surfaces:

Problem 2.3. *Let M, $M' \subset \mathbf{R}^3$ be smooth orientable closed surfaces. Suppose there exists a diffeomorphism $f\colon M \to M'$ which preserved the Gauss curvature and the Gauss map. Does it follow then that M and M' are congruent?*

This problem also has a discrete version:

Problem 2.4. *Let P, $P' \subset \mathbf{R}^3$ be polyhedral surfaces. Suppose that the faces of P and P' are parallel and have the same area. Does it follow then that P and P' are congruent?*

3. Unfoldability of Convex Polyhedra

A well-known problem [46, 142, 145, 195], which may be traced back to the Renaissance artist Albrecht Dürer [49], is concerned with cutting a convex polyhedral surface P along some spanning tree T of its edges so that it may be isometrically embedded, or unfolded without overlaps, into the plane. If P admits such a one-to-one unfolding (for some choice of T) then we say that P is *unfoldable*.

Problem 3.1 (Dürer [49], 1525; Shephard [170], 1975)**.** *Is every convex polytope unfoldable?*

The author has shown [70] that this is always possible after an affine transformation of the surface. In particular, unfoldability of a convex polyhedron does not depend on its combinatorial structure, which settles a problem of Croft, Falconer, and Guy [41, B21]. To describe this result more explicitly, assume that the polyhedron P is in *general position* with respect to L, by which we mean that for a unit vector u parallel to L the height function $\langle\cdot, u\rangle$ has a unique maximizer and a unique minimizer among the vertices of P. There is an open dense set of such lines in the real projective space $\mathbf{RP}^2$. In [70] it is shown that P becomes unfoldable after a sufficiently large stretching in any such direction. To describe what we mean by *stretching* it would be convenient to assume, after a rotation, that L is parallel to the z-axis in $\mathbf{R}^3$, then our transformation is given by $(x, y, z) \longmapsto (x, y, \lambda z)$. This will have the effect of making P arbitrarily "thin" or "needle-shaped". Thus, roughly speaking, needle-shaped convex polytopes are unfoldable.

Dürer's problem is the most famous question in the theory of foldings and unfoldings [46], or origami research, which has had wide applications ranging from assembling satellite dishes in space to the expansion of implanted stents in human arteries. For extensive backgroud on this problem, and an outline of the author's solution in the affine case, see

people.math.gatech.edu/∼ghomi/Talks/durerslides.pdf

Yet there is still no algorithm for unfolding a convex polytope injectively despite intense efforts by computer scientists. The difficulty of Dürer's problem is that it is essentially an intrinsic question, yet there is no known intrinsic method to detect the edges of a convex polyhedron. Indeed, Alexandrov's embedding theorem for convex surfaces [9]—which states that any locally convex polyhedral metric on $\mathbf{S}^2$ may be realized as a convex polyhedron in $\mathbf{R}^3$—is not constructive and gives no hint as to which geodesics between a pair of vertices are realized as edges; see also Pak [145]. In 2008, a more constructive proof was given by Bobenko and Izmestiev [24]; however, this proof does not specify the location of the edges either. Thus Dürer's problem provides another opportunity to deepen our understanding of isometric embeddings.

Problem 3.2. *Does there exist a reasonably simple algorithm for detecting the edges of a convex polyhedron intrinsically?*

The edge graph of P is not the unique graph in P whose vertices coincide with those of P, whose edges are geodesics, and whose faces are convex. It seems reasonable to expect that Dürer's conjecture should be true if and only if it holds for this wider class of *pseudo edge graphs*. This approach was studied by Tarasov [180] in 2008, who announced some negative results in this direction; however, Tarasov's paper has not been published. Checking the validity of Tarasov's constructions would be an important step towards solving Dürer's problem:

Problem 3.3. *Does there exist a convex polyhedron with a pseudo edge graph which is not unfoldable.*

The earliest known examples of simple edge unfoldings for convex polyhedra are due to Dürer [49], although the problem which bears his name was first formulated by Shephard [170]. Furthermore, the assertion that a solution can always be found, which has been dubbed Dürer's conjecture, appears to have been first published by Grünbaum [85, 86]. There is empirical evidence both for and against this supposition. On the one hand, computers have found simple edge unfoldings for countless convex polyhedra through an exhaustive search of their spanning edge trees. On the other hand, there is still no algorithm for finding the right tree [164, 118], and computer experiments suggest that the probability that a random edge unfolding of a generic polyhedron overlaps itself approaches 1 as the number of vertices grow [163]. General cut trees have been studied at least as far back as Alexandrov [9] who first established the existence of simple unfoldings (not necessarily simple *edge* unfoldings) for all convex polyhedra, see also [99, 131, 44] for recent related results. Other references and background may be found in [46].

4. Area and Volume of Convex Surfaces

Problem 4.1 (A. D. Alexandrov [8])**.** *Of all convex surfaces with a fixed intrinsic diameter, is the one with the greatest area a doubled disk?*

It is known that the answer to the above question is affirmative for surfaces of revolution [121] and that in the class of tetrahedra the maximizer is, rather surprisingly, the regular tetrahedron [119]. See also [120] for results relating the intrinsic and extrinsic diameter of convex surfaces. For other partial results using techniques in Riemannian geometry see [172, 161]

Problem 4.2 (Volume of surfaces of constant width)**.** *Let $S \subset \mathbf{R}^3$ be a closed surface of constant width and fixed area. How small can the volume of S be?*

The above problem has been solved in $\mathbf{R}^2$: the Reuleaux triangle has the least area among all closed curves of constant width. For a recent proof of this result, which is originally due to Blaschke and Lebesque, see Harrell [95]. For other references see [42].

Problem 4.3 (Surfaces with strips of constant area)**.** *Let $S \subset \mathbf{R}^3$ be a closed surface of diameter d. Suppose that there exists a constant $h < d$ so that whenever a pair of planes separated by a distance of h intersect S, the area of S contained between these planes is constant. Does it then follow that S is a sphere?*

5. Extremal Problems for Space Curves

What is the smallest length of wire which can be bent into a shape that never falls through the gap behind a desk? What is the shortest orbit which allows a satellite to survey a spherical asteroid? These are well-known open problems [193, 143, 42, 96] in classical geometry of space curves $\gamma\colon [a,b] \to \mathbf{R}^3$, which are concerned with minimizing the length L of γ subject to constraints on its *width* w and *inradius* r respectively. Here w is the infimum of the distances between all pairs of parallel planes which bound γ, while r is the supremum of the radii of all spheres which are contained in the convex hull of γ and are disjoint from γ. In 1994–1996 Zalgaller [192, 193] conjectured four explicit solutions to these problems, including the cases where γ is restricted to be closed, i.e., $\gamma(a) = \gamma(b)$. Recently the author has been able to employ a combination of integral geometric and topological techniques to confirm Zalgaller's conjectures [71] between 83% and 99% of their stated value, while he has also found a counterexample to one of them, but sharp answers have not yet been found.

Problem 5.1 (Zalgaller [192] 1994)**.** *What is the shortest curve in $\mathbf{R}^3$ with a given width or inradius?*

The author has shown that, for any rectifiable curve $\gamma\colon [a,b] \to \mathbf{R}^3$,

$$\frac{L}{w} \geq 3.7669. \tag{2}$$

Furthermore if γ is closed,

$$\frac{L}{w} \geq \sqrt{\pi^2 + 16} > 5.0862. \tag{3}$$

In [192] Zalgaller constructs a curve, "L_3", with $L/w \leq 3.9215$. Thus (2) is better than 96% sharp (since $3.7669/3.9215 \geq 0.9605$). Further, there exists a closed cylindrical curve with $L/w < 5.1151$, which shows that (3) is at least 99.43% sharp. In particular, the length of the shortest closed curve of width 1 is approximately 5.1.

The author has also found estimates for the inradius problem. Obviously $w \geq 2r$, and thus the above inequalities immediately yield $L/r \geq 7.5338$ for general curves, and $L/r \geq 10.1724$ for closed curves. Using different techniques, however, these estimates may be improved as follows: for any rectifiable curve $\gamma\colon [a, b] \to \mathbf{R}^3$,

$$\frac{L}{r} \geq \sqrt{(\pi + 2)^2 + 36} > 7.9104. \tag{4}$$

Furthermore if γ is closed,

$$\frac{L}{r} \geq 6\sqrt{3} > 10.3923. \tag{5}$$

In [194, Sec. 2.12] Zalgaller constructs a spiral curve with $L/r \leq 9.5767$, which shows that (4) is better than 82.6% optimal. Further, in [193], he produces a curve composed of four semicircles with $L/r = 4\pi$; see also [143] where this "baseball stitches" curve is rediscovered in 2011. Thus we may say that (5) is better than 82.69% optimal.

Both the width and inradius problems may be traced back to a 1956 question of Bellman [20] motivated by harmonic analysis: how long is the shortest escape path for a random point (lost hiker) inside an infinite parallel strip (forest) of known width? See [56] for more on these types of problems. Our width problem is the analogue of Bellman's question in $\mathbf{R}^3$. The inradius problem also has an intuitive reformulation known as the "sphere inspection" [194, 143] or the "asteroid surveying" problem [31]; see

mathoverflow.net/questions/69099/shortest-closed-curve-to-inspect-a-sphere

To describe this variation, let us say that a space curve γ *inspects* the sphere $\mathbf{S}^2$, or is an *inspection curve*, provided that γ lies outside $\mathbf{S}^2$ and for each point x of $\mathbf{S}^2$ there exists a point y of γ such that the line segment xy does not enter $\mathbf{S}^2$ (in other words, x is "visible" from y). It is easy to see that γ inspects $\mathbf{S}^2$, after a translation, if and only if its inradius is 1 [193, p. 369]. Thus finding the shortest inspection curve is equivalent to the inradius problem for $r = 1$.

Problem 5.2 (Volume of the convex hull of closed curves)**.** *Let* Γ *be a closed curve of fixed length* L *in* $\mathbf{R}^3$*. How big can the volume of the convex hull of* Γ *be.*

For a partial result under symmetry conditions for the above problem see [130]. The above problem has been solved in Euclidean spaces of even dimensions [168]. Also the problem is solved for open arcs in $\mathbf{R}^3$ [51]. For some other related results and questions see [193].

Problem 5.3 (Area of the convex hull of closed curves)**.** *Let Γ be a closed curve of fixed length L in $\mathbf{R}^3$, and A be the area of the convex hull of Γ. Show that A is biggest when Γ is a circle, in which case we consider the convex hull of Γ as a doubly covered disk.*

If Γ is a simple curve which lies on the boundary of its convex hull, then it divides the boundary of the convex hull into a pair of disks each of which have zero curvature. This observation together with the isoperimetric inequality for surfaces of nonpositive curvature, first proved by Andre Weil, may be used to solve the above problem in the case where Γ is simple and lies on the boundary of its convex hull.

6. Minimal and CMC Surfaces

Problem 6.1 (Meeks [125, 129, 126], 1978)**.** *Is every compact connected minimal surface bounded by a pair of convex planar curves topologically an annulus?*

An affirmative solution to the above problem would lead to a generalization of Shiffman's classical theorem [171] on level set of minimal annuli. Partial or related results have been obtained in [167, 127, 128, 138, 155, 104, 52, 62]. See the article by Hoffman and Meeks [97] for a nice introduction to the conjecture.

Problem 6.2 (Earp, Fabiano, Meeks, and Rosenberg [50], 1991)**.** *Does there exist an embedded compact surface of constant mean curvature which is bounded by a circle, but is not a piece of a sphere.*

If the surface is only immersed, not embedded, then the answer to the last problem is yes, as was shown by Kapouleas [106]. Furthermore [11] Alias, Lopez, and Palmer showed that the answer is yes, if the surface is assumed to be stable and a topological disk.

Problem 6.3 (Ros and Rosenberg [156], 1996)**.** *Show that any compact embedded CMC surface which is bounded by a convex planar curve, and lies on one side of the boundary plane, is topologically a disk.*

The above problem has been affirmatively solved by Barbosa and Jorge assuming that the surface is stable [18]. For more background and references related to the last problem see the book of Lopez [117].

7. Negatively Curved Surfaces

The next problem is due to John Milnor [132], and would generalize famous theorems of Hilbert and Efimov. Hilbert showed that there exists no complete surfaces of constant negative curvature in $\mathbf{R}^3$, and Efimov proved that there exists no complete surfaces of negative curvature in $\mathbf{R}^3$ whose curvature is bounded away from zero. Proof of Hilbert's theorem may be found in many elementary texts on differential geometry. For Efimov's proof see [132]. Some relatively recent proofs of these results have been announced in [152].

Problem 7.1 (Milnor [132], 1972)**.** *Are there any complete surfaces of negative curvature in Euclidean 3-space whose principal curvatures are bounded away from zero?*

An even older problem on nagatively curved surfaces is the following:

Problem 7.2 (Hadamard [90, 27, 159], 1898)**.** *Are there any complete surfaces of negative curvature in Euclidean 3-space whose principal curvatures are bounded away from zero?*

In the immersed case, this conjecture was settled by Nadirashvili [136] who constructed complete minimal negatively curved surfaces in a ball, see also [105, 36, 123, 116]; however, such surfaces cannot be embedded, since by a result of Colding and Minicozzi [35], an embedded minimal surface must be unbounded. An approach for constructing a possible counterexample to the above conjecture has been outlined by Rozendorn [159].

Problem 7.3. *Does there exist any complete negatively curved surfaces with negative Euler characteristic contained in between a pair of parallel planes in* $\mathbf{R}^3$.

It is not difficult to construct such surfaces with nonnegative Euler characteristic, see the author's paper with Chris Connell [37, Note 1.4]. See also [38] and [30] for relevant results on topology of negatively surfaces.

8. Umbilic Points

Problem 8.1 (Carathéodory, 1922)**.** *Show that every closed convex surface in* $\mathbf{R}^3$ *has at least two umbilic points.*

According to Struik [179], the earliest references to the conjecture attributed to Carathéodory appear in the works of Cohn-Vossen, Blaschke, and Hamburger dating back to 1922. The first significant results on the conjecture were due to Hamburger, who established the analytic case in a series of long papers [91, 92, 93] published in 1940–41. Attempts to find shorter proofs attracted the attention of Bol [25], Klotz [110], and Titus [182] in the ensuing decades. As late as 1993, however, Scherbel [162] was still correcting some errors in these simplifications, while reconfirming the validity of Hamburger's theorem. Another reexamination of the proof of the analytic case appears in a comprehensive paper of Ivanov [102] who supplies his own arguments for clarifying various details. All the works mentioned thus far have been primarily concerned with establishing the analytic version of a local conjecture attributed to Loewner, which states that

Problem 8.2 (Loewner)**.** *Show that the index of any singularity of a principal line fields on a surface is at most one.*

A positive resolution of Loewner's conjecture would imply Carathéodory's conjecture via Poincaré-Hopf index theorem. See Smyth and Xavier [173, 174, 175] for studies of Loewner's conjecture in the smooth case, and Lazarovici [114] for a global result on principal foliations. Another global result is by Feldman [55], who showed that generic closed convex surfaces have four umbilics; also see [74] for some applications of the h-principle to studying homotopy classes of principal lines. A global generalization of Carathéodory's conjecture is discussed in [57]. There is also an interesting analogue of the conjecture for noncompact complete convex hypersurfaces:

Problem 8.3 (Toponogov [183], 1995)**.** *Let M be a complete noncompact convex surface in $\mathbf{R}^3$, with principal curvatures k_1, k_2, then show that $\inf_M |k_1 - k_2| = 0$.*

In other words, Toponogov suggests that every complete noncompact convex surface must have an umbilic "at infinity". A number of approaches to proving the Carathéodory or Loewner conjecture in the smooth case are discussed in [144, 139, 88], and more references or background may be found in [176, 89]. In recent joint work with Ralph Howard [73], the author has found some real evidence in support of this conjecture. At the same time we constructed a number of surfaces which may be considered counterexamples in a weak sense. A number of approaches to proving the Carathéodory or Loewner conjecture in the smooth case are discussed in [144, 139, 88]. In particular Guilfoyle and Klingenberg [88] announced a solution to the conjecture in 2008, although that work does not seem to have been published yet. More references or background may be found in [176, 89].

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