A variant of Kuperberg’s proof of the Bourgain-Milman theorem

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Some points in the slides require additional explanation, either because what is written is not literally true as it stands, or is not obvious.

These points are indicated by a ’(!)’ and will be explained verbally during the lecture.
Let $K$ be a convex body in $\mathbb{R}^n$.

$$K^\circ := \{\xi; \xi \cdot x \leq 1; x \in K\}.$$ 

The Mahler volume of $K$ is

$$M(K) = |K|\|K^\circ|.$$ 

\textit{Kuperberg’s theorem:}

Theorem

$$M(K) \geq \pi^n/n!$$

if $K$ is symmetric.
Scheme of Kuperberg’s proof: Define a quantity $Q(K)$ (‘energy’ or ‘Gauss linking integral’) such that

$$M(K) \geq Q(K)$$

and

$$Q(K) \geq \frac{\pi^n}{n!}.$$

Strangely, $Q(K)$ is minimized when $K$ is a ball.
Variant: Let $\phi$ be a convex function on $\mathbb{R}^n$. The Legendre transform is defined as

$$
\phi^*(\xi) = \sup_x \xi \cdot x - \phi(x).
$$

**Theorem**

$$
\int e^{-\phi} \int e^{-\phi^*} \geq \pi^n,
$$

if $\phi$ is symmetric.
Scheme of proof: Define a quantity $Q(\phi)$ such that

$$M(\phi) := \int e^{-\phi} \int e^{-\phi^*} \geq Q(\phi)$$

and

$$Q(\phi) \geq \pi^n.$$ 

$Q(\phi)$ is minimized for $\phi = \phi_0 = x^2/2$.

What is $Q(\phi)$?
Let
\[ \lambda = \{(x, \xi); \xi = \partial \phi(x)/\partial x\} \subset \mathbb{R}^{2n}. \]
Let
\[ \Lambda = \Lambda = \lambda \times \lambda = \{(x, \xi, y, \eta); \xi = \partial \phi(x)/\partial x, \eta = \partial \phi(y)/\partial y\}. \]
Now write \( z = x + iy, \zeta = \xi + i\eta \) and let
\[ \Omega = ((i/2) \sum dz_j \wedge d\bar{\zeta}_j)^n/n! = \omega^n/n!. \]
\( \omega \) can be seen as a holomorphic symplectic form on \( \mathbb{C}^{2n} \). (!)
We first put
\[ I(\phi) = \int_{\Lambda} e^{-(1/2)z \cdot \bar{\zeta}} \Omega. \]
Then we let
\[ Q(\phi) = 2^{-n} \int_{\Lambda} |e^{-(1/2)z \cdot \bar{\zeta}} \Omega| = 2^{-n} \int_{\Lambda} e^{-(1/2)(\xi \cdot x + \eta \cdot y)} |\Omega| \]
\[ |I(\phi)| \leq \int_{\Lambda} e^{-1/2(\xi \cdot x + \eta \cdot y)} |\Omega| = 2^n Q(\phi). \]

The point is that \( I(\phi) \) is independent of \( \phi \) since it is the integrand of a closed form. (!) Take \( \phi = \phi_0 = x^2 / 2 \). Then \( \Lambda = \{ z = \zeta \} \), and

\[ \Omega = (i/2 \sum dz_j \wedge d\bar{z}_j)^n / n! = dm, \]

volume form on \( \mathbb{C}_z^n \). Hence

\[ I(\phi) = I(\phi_0) = \int_{\mathbb{C}_z^n} e^{- (1/2)|z|^2} dm = (2\pi)^n. \]

Hence \( O(\phi) \geq \pi^n \).
It remains to prove the estimate from above of $Q$,

$$Q(\phi) \leq M(\phi).$$

Recall

$$\phi^*(\xi) = \sup_x \xi \cdot x - \phi(x), \quad \text{eq. for } \xi = \partial \phi(x).$$

Hence

$$\phi(x) + \phi^*(\xi) = \xi \cdot x, \quad \text{on } \Lambda, \quad \text{and} \quad \phi(y) + \phi^*(\eta) = \eta \cdot y.$$ 

Let

$$\pi : \Lambda \to \mathbb{R}^{2n}_{ts}, \quad t = (x + y)/2, \quad s = (\xi - \eta)/2.$$ 

**Lemma**

$\pi$ is injective, and surjective if $\phi$ grows faster than any linear function.
To prove the lemma, let for \( t \) fixed.

\[
A_t = \{ x + y = 2t \}.
\]

Put

\[
\Phi(x) = \phi(x) + \phi(2t - x).
\]

Then

\[
\frac{\partial \Phi}{\partial x} = \frac{\partial \phi(x)}{\partial x} - \frac{\partial \phi(y)}{\partial y} = \xi - \eta.
\]

Hence injective if \( \phi \) is strictly convex and surjective if \( \phi \) grows faster than linearly.
Pulling back the Mahler integral to $\Lambda$ we get

$$M(\phi) = \int_{\mathbb{R}^{2n}} e^{-\phi(t)+\phi^*(s)} dt ds = \int_{\Lambda} e^{-(\phi+\phi^*) \circ \pi \pi^* (dt ds)}.$$ 

**Lemma**

$$\pi^*(dt ds) = 2^{-n}|\Omega|.$$ 

Accepting this we get

$$M(\phi) = 2^{-n} \int_{\Lambda} e^{-(\phi((x+y)/2)+\phi^*((\xi-\eta)/2))}|\Omega| \geq$$

$$2^{-n} \int_{\Lambda} e^{-(1/2)(\phi(x)+\phi(y)+\phi^*(\xi)+\phi^*(\eta))}|\Omega| =$$

$$2^{-n} \int_{\Lambda} e^{-(1/2)(x \cdot \xi + y \cdot \eta)}|\Omega| = Q(\phi).$$
It remains to prove that

$$\pi^*(dt ds) = 2^{-n}|\Omega|.$$ 

Recall that $\Omega = \omega^n/n!$,

$$\omega = (i/2) \sum dz_j \wedge d\bar{\zeta}_j = (i/2)(\sum dx_j \wedge d\xi_j + dy_j \wedge d\eta_j + i \sum dy_j \wedge d\xi_j - dx_j \wedge d\eta_j).$$

Put $\tau = \sum dt_j \wedge ds_j; dt ds = \pm \tau^n/n!$. Then

$$\pi^*(\tau) = (1/4) \sum (dx_j + dy_j) \wedge (d\xi_j - d\eta_j) = (1/4) \sum dx_j \wedge d\xi_j - dy_j \wedge d\eta_j + dy_j \wedge d\xi_j - dx_j \wedge d\eta_j).$$

Compare and use $\sum dx_j \wedge d\xi_j = \sum dy_j \wedge d\eta_j = 0. (!)$
Remarks

As we said $\omega$ is a holomorphic symplectic form on $\mathbb{C}^{2n}$. Its real and imaginary parts are both real symplectic forms. The real part vanishes on $\Lambda$. The imaginary part is a symplectic form on $\Lambda$, i.e. nondegenerate there. The point of the main lemma is that $x + y = 2t$ and $\xi - \eta = 2s$ are Darboux coordinates on $\Lambda$; they transform the imaginary part to the standard symplectic form on $\mathbb{R}^{2n}$. 
Comments on Nazarov’s proof

Nazarov considers Bergman spaces of the form

\[ A^2_K = \{ f \in H; \int_{x \in K} |f(x + iy)|^2 dx dy < \infty \}. \]

The Bergman kernel for such a space is

\[ B(z) = \sup_f |f(z)|^2 / \| f \|^2. \]

His main technical result is

**Theorem**

\[ B(0) \geq c^n |K|^{-2}. \]
The main difficulty in an estimate of the Bergman kernel from below is that one needs to construct a function $f$ which has a large value at a point compared to its norm. He uses Hörmander’s $L^2$-estimates for $\bar{\partial}$.

The next step is to couple the thm with an estimate from above (which is more elementary)

$$B(0) \leq \pi^n |K^\circ|/|K|.$$ 

The result is

$$c^n |K|^{-2} \leq B(0) \leq \pi^n |K^\circ|/|K|,$$

which gives the BM-theorem.
More generally, we can consider Bergman spaces defined by a convex function $\phi$

$$A^2_\phi = \{ f \in H; \int |f(x + iy)|^2 e^{-\phi(x)} \, dx \, dy < \infty \}.$$ 

The analog of the upper estimate is then

$$B(0) \leq \pi^n \frac{\int e^{-\phi^*}}{\int e^{-\phi}}.$$
\[ A_{\phi+\psi}^2 = \{ f \in H; \int |f(x + iy)|^2 e^{-(\phi(x)+\psi(x))} \, dx \, dy < \infty \}. \]

Let

\[ B_{\phi,\psi}^2 = \{ f \in H; \int |f(x + iy)|^2 e^{-(\phi(x)+\psi(y))} \, dx \, dy < \infty \} \]

and let \( B' \) be the Bergman kernel for the second space.

**Theorem**

\[ B'(0) \leq C^n B(0). \]
The main interest of the theorem is that $B'(0)$ is very easy to estimate from below, since $f = 1$ lies in $B^2_{\phi, \psi}$ (but not in $A^2_{\phi + \psi}$).

$$B'(0) \geq \left( \int e^{-\phi} \int e^{-\psi} \right)^{-1}.$$ 

Hence, with $\psi = \phi$,

$$\left( \int e^{-\phi} \int e^{-\phi} \right)^{-1} \leq B'(0) \leq C^n B(0) \leq C_1^n \int e^{-\phi^*} \int e^{-\phi},$$

which gives BM-theorem again.
One instance of the thm is easy to see directly: If $\psi$ is a quadratic form, e.g. $\psi(x) = x^2$.

Since $x^2 - y^2 = \text{Re} z^2$ one finds that $B(0) = B'(0)$. (!)

I do not know if one can take $C = 1$ in the theorem. If so, the above argument gives the same bound as Kuperberg’s.
The proof of the theorem uses the family of spaces

\[ A_s^2 = \{ f \in H; \int |f(x + iy)|^2 e^{-(\phi(z) + \psi(sz))} \, dx \, dy < \infty \}. \]

Here \( \phi(z) = \phi(\text{Re} \, z) \) and \( s \) is a complex parameter.

Since \( \phi(z) + \phi(sz) \) is plurisubharmonic in \((s, z)\) it follows from an earlier result of mine that the logarithm of the Bergman kernel \( \log B_s(0) \) is subharmonic in \( s \).

One can therefore estimate \( B'(0) = B_{-i} \) by \( B_s(0) \) for \( s \) real by the Poisson integral representation.