Floating bodies and random polytopes.

Olivier Guédon

LAMA, Université Gustave Eiffel

May 19, 2020
Geometry of convex bodies

Let $X$ be a random vector in $\mathbb{R}^n$, $X_1, \ldots, X_N$ independent copies of $X$. We study

$$\text{absconv}(X_1, \ldots, X_N) = AB_1^N \subset \mathbb{R}^n$$

where $A$ is a matrix which columns are the vectors $X_i$. 

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2. geometry: asymptotic behavior as $N \geq n$ et $n \to \infty$: extremal properties of the volume of the polytope or its polar.
3. probability: geometric properties of the polytope according to the law of the random vector which generates the polytope, properties of the operator norm of $A$. 
Random polytopes

A key result in the local theory of Banach spaces (due to Gluskin in 1981): the Banach Mazur distance between 2 such random polytopes is “extremal”: $X \sim \mathcal{N}(0, \text{Id})$, with high probability, for $N \geq 10n$

$$d(P_N, P'_N) \geq cn$$
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3. Extremal properties of such random polytopes: (CFPP 2015) if \( X \) has a bounded density (by 1) then

\[
\mathbb{E} \text{Vol}(\text{absconv}(X_1, \ldots, X_N))^o
\]

is maximal when \( X \sim \mathcal{U}_{B_2^n} \).
Random matrices : $\Gamma = (X_1, \ldots, X_N)^T : \ell_2^n \rightarrow \ell_2^N$

Study of the extreme singular values of the matrix :

$$s_1(\Gamma) = \sup_{|x|_2=1} |\Gamma x|_2 = \sup_{|x|_2=1} \left( \sum_{j=1}^N \langle X_j, x \rangle^2 \right)^{1/2}$$

$$s_N(\Gamma) = \inf_{|x|_2=1} |\Gamma x|_2 = \inf_{|x|_2=1} \left( \sum_{j=1}^N \langle X_j, x \rangle^2 \right)^{1/2}$$

By duality, showing that $s_N \geq \alpha \sqrt{N}$ is equivalent to

$$\alpha \sqrt{NB_2^n} \subset AB_2^N (\subset \sqrt{NP_N})$$

LPRT (2005), LPRTV (2006) : good hypotheses on the random vector $X$. Net arguments. In all these arguments, they need a good bound on $s_1$ to get a lower bound on $s_N$.

Kolesnikov, Mendelson (2014)
### Question

$X$ a random vector in $\mathbb{R}^n$, $X_1, \ldots, X_N$ independent copies of $X$. 

---

**Definition**

A family of floating bodies. Let $X$ be a symmetric random vector, for every $p \geq 1$, set $K_p(X) = \{ t \in \mathbb{R}^n, \mathbb{P}(\langle X, t \rangle \geq 1) \leq e^{-p} \}$. 

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### Question

X a random vector in \( \mathbb{R}^n \), \( X_1, \ldots, X_N \) independent copies of X.

Define a "natural set" \( K \) associated to \( X \) such that with high probability

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K \subset \text{absconv}(X_1, \ldots, X_N)
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Floating body and geometry of the polytopes.

Question

1. Define a ”natural set” $K$ associated to $X$ such that with high probability

   $$K \subset \text{absconv}(X_1, \ldots, X_N)$$

2. Give some kind of precise description of $K$?
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Assume $X$ is reasonably "nice"

- For every $t \in \mathbb{R}^n$, $\langle X, t \rangle$ has moments of all order. And define

$$B(L_p(X)) = \left\{ t \in \mathbb{R}^n, (\mathbb{E}|\langle X, t \rangle|^p)^{1/p} \leq 1 \right\}$$

Then by Chebychev inequality

$$\frac{1}{e} B(L_p(X)) \subset K_p(X)$$
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$$B(L_p(X)) = \left\{ t \in \mathbb{R}^n, (\mathbb{E}|\langle X, t \rangle|^p)^{1/p} \leq 1 \right\}$$

- Assume also that there exists $D \geq 1$ such that

$$\forall q \geq 2, \forall t \in \mathbb{R}^n, \left( \mathbb{E}|\langle X, t \rangle|^{2q} \right)^{1/2q} \leq D \left( \mathbb{E}|\langle X, t \rangle|^q \right)^{1/q}$$

Then by Paley-Zygmund,

$$K_p(X) \subset 2B(L_{c_1p}(X))$$

where $c_1$ depends only on $D$. 

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**Conclusion**

\[ \exists C_1(D) \geq 1, \forall p \geq 2, \quad \frac{1}{e} B(L_p(X)) \subset K_p(X) \subset C_1 B(L_p(X)) \]
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### Conclusion

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\exists C_1(D) \geq 1, \forall p \geq 2, \quad \frac{1}{e} B(L_p(X)) \subset K_p(X) \subset C_1 B(L_p(X)),
\]

### Remark

The polar $B(L_p(X))^o$ is called the $Z_p$-centroid body of $X$

\[
Z_p(X) = B(L_p(X))^o
\]

and is well studied in the geometry of log-concave measures.
Floating body - Various examples.

Set $G \sim \mathcal{N}(0, \text{Id})$ then

$$K_p(G) \approx \frac{1}{\sqrt{p}} B_2^n$$ and $$K_p(G)^{o} \approx \sqrt{p} B_2^n$$
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- Set $X$ uniformly distributed on a symmetric convex body $K$ then
  
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Theorem of Montgomery-Smith (1990)

$$\mathbb{P} \left( \sum_{i=1}^{n} x_i \varepsilon_i > K_{1,2}(x, t) \right) \approx e^{-ct^2}$$

where

$$K_{1,2}(x, \sqrt{p}) = \sum_{i=1}^{p} x_i^* + \frac{1}{\sqrt{p}} \left( \sum_{i=p+1}^{n} x_i^{*2} \right)^{1/2}$$
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  where $h_{Z_p(X)}(\theta) = (\mathbb{E}\langle X, \theta \rangle^p)^{1/p}$.

- Set $\mathcal{E} = (\varepsilon_1, \ldots, \varepsilon_n)$ where $\varepsilon_i$ are iid Rademacher r.v. then
  \[ K_p(\mathcal{E}) \approx \text{conv} \left( B_1^n \cup \frac{1}{\sqrt{p}} B_2^n \right) \quad \text{and} \quad K_p(\mathcal{E})^o \approx B_\infty^n \cap \sqrt{p} B_2^n \]
The result (GKKMR 2019).

1. Norm: $\| \cdot \|$ is a norm.

2. Small ball property: There exist $\gamma > 0$ and $\delta > 0$ such that for all $t \in \mathbb{R}^n$, $P(\langle X, t \rangle \geq \gamma \|t\|) \geq \delta$.

3. Moment assumption: For some $r > 0$ and $L > 0$, we have for all $t \in \mathbb{R}^n$, $E|\langle X, t \rangle|^r \leq L \|t\|^r$.

Theorem: Let $0 < \alpha < 1$, $p = \alpha \log(eN^n)$ and $N \geq c_0(\alpha, r, \delta, L/\gamma) n$. Therefore, with probability $\geq 1 - 2 \exp(-CN^{1-\alpha n})$, $K_0 \subset \text{absconv}(X_1, \ldots, X_N)$.
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$$\frac{1}{2} K_p^o \subset \text{absconv}(X_1, \ldots, X_N)$$
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$$c_2 \sqrt{\alpha \log \left(\frac{eN}{n}\right)} B_2^n \subset \text{absconv}(X_1, \ldots, X_N)$$
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- (LPRT 2005) $\mathcal{E} = (\varepsilon_1, \ldots, \varepsilon_n)$ where $\varepsilon_i$ are iid Rademacher r.v. then

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- (DGT 2009) $X$ uniformly distributed on a symmetric convex body $K$ then
  
  $c_2 Z_p(X) \subset \text{absconv}(X_1, \ldots, X_N)$
The case of $q$-stable random vector.

$X = (\xi_1, \ldots, \xi_n)$ with $\xi_i$ iid $q$-stable: $\mathbb{E} \exp(itX) = \exp(-|t|^q/2)$

Observe that $\langle X, t \rangle = \sum t_i \xi_i \sim |t|^q \xi$ and remember that for every large enough $u$

$$\mathbb{P}(\xi \geq u) \approx 1/u^q$$
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Theorem

For all $q \geq 1$, taking $p = \alpha \log(eN/n)$, we have

$$c_2(q) \left( \frac{N}{n} \right)^{\alpha/q} B_n^q \subset K_p(X)^o \subset 2\text{absconv}(X_1, \ldots, X_N)$$
II) Stochastic domination and floating bodies.

Definition

Let $X$ and $Y$ be two centered random vectors in $\mathbb{R}^n$. We say that $X$ dominates $Y$ when there exist $\lambda_1$ and $\lambda_2$ such that

$$\forall t \in \mathbb{R}^n, \forall u \in \mathbb{R}, \quad P(\langle X, t \rangle \geq u) \geq \lambda_1 P(\langle Y, t \rangle \geq \lambda_2 u)$$

This gives

$$K_p(X) \subset \lambda_2 K_{p'}(Y)$$

with $p' = p - \log(1/\lambda_1)$.

This property is stable by tensorization: if $x$ and $y$ are symmetric r.v. such that for every $u > 0$, $P(x > u) \leq \lambda_1 P(y > \lambda_2 u)$ then $X = (x_1, \ldots, x_n)$ dominates $Y = (y_1, \ldots, y_n)$ with constants $c_1 \lambda_1$ and $c_2 \lambda_2$, where $x_1, \ldots, x_n$ are iid copies of $x$ and $y_1, \ldots, y_n$ are iid copies of $y$. 
Let $X = (\xi_1, \ldots, \xi_n)$ with $\xi_i$ independent copies of a symmetric r.v. $\xi$. Assume $\mathbb{E}\xi^2 = 1$ and $\mathbb{P}(|\xi| \geq \gamma_0) \geq \delta_0$. 

By tensorisation, for $X = (\xi_1, \ldots, \xi_n)$ and $E = (\varepsilon_1, \ldots, \varepsilon_n)$, we get that there exist $\lambda_1, \lambda_2$ such that for every $t \in \mathbb{R}^n$ $\forall u \in \mathbb{R}$, 

$\mathbb{P}(\langle X, t \rangle \geq u) \geq \lambda_1 \mathbb{P}(\langle E, t \rangle \geq \lambda_2 u)$ 

In conclusion, $K^p(X) \subset \lambda_2 K^p(E)$ and $K^p(X) \supset \lambda_2^{-1} K^p(E)$ where $p' = p - \log(1/\lambda_1)$.
Let $X = (\xi_1, \ldots, \xi_n)$ with $\xi_i$ independent copies of a symmetric r.v. $\xi$. Assume $\mathbb{E}\xi^2 = 1$ and $\mathbb{P}(|\xi| \geq \gamma_0) \geq \delta_0$. Then for every $u \in \mathbb{R}$

$$\mathbb{P}(\xi \geq u) \geq \delta_0 \mathbb{P}(\varepsilon \geq \frac{u}{\gamma_0})$$
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By tensorisation, for $X = (\xi_1, \ldots, \xi_n)$ and $\mathcal{E} = (\varepsilon_1, \ldots, \varepsilon_n)$, we get that there exist $\lambda_1, \lambda_2$ such that for every $t \in \mathbb{R}^n$

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$$
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$$

By tensorisation, for $X = (\xi_1, \ldots, \xi_n)$ and $\mathcal{E} = (\varepsilon_1, \ldots, \varepsilon_n)$, we get that there exist $\lambda_1, \lambda_2$ such that for every $t \in \mathbb{R}^n$

$$
\forall u \in \mathbb{R}, \quad \mathbb{P}(\langle X, t \rangle \geq u) \geq \lambda_1 \mathbb{P}(\langle \mathcal{E}, t \rangle \geq \lambda_2 u)
$$

In conclusion, $K_p(X) \subset \lambda_2 K_{p'}(\mathcal{E})$ and $K_p(X) \supset \lambda_2^{-1} K_{p'}(\mathcal{E})$ where

$$
p' = p - \log(1/\lambda_1).
$$

**Theorem (GLT 2018)**

Let $X = (\xi_1, \ldots, \xi_n)$ with $\xi_i$ indep. copies of $\xi$. Suppose that $\mathbb{E}\xi^2 = 1$ and $\mathbb{P}(|\xi| \geq \gamma) \geq \delta$. Then for $N \geq c_0(\alpha, \gamma, \delta)n$, we have with proba $\geq 1 - 2 \exp(-c_1 N^{1-\alpha}n^\alpha)$,

$$
\text{absconv}(X_1, \ldots, X_N) \supset c_2 \left(B_{\infty}^n \cap \sqrt{\alpha \log \left(\frac{eN}{n}\right)} B_2^n\right)
$$
Theorem

Let \( X = (\xi_1, \ldots, \xi_n) \) be an unconditional random vector in \( \mathbb{R}^n \). Assume that there exist \( \gamma \) and \( \delta > 0 \) such that for any \( i = 1, \ldots, n \)

\[
\mathbb{P}(|\xi_i| \geq \gamma) \geq \delta
\]

then

\[
K_p(X) \subset \frac{c(\delta)}{\gamma} K_p(\mathcal{E})
\]
Theorem

Let $X = (\xi_1, \ldots, \xi_n)$ be an unconditional random vector in $\mathbb{R}^n$. Assume that there exist $\gamma$ and $\delta > 0$ such that for any $i = 1, \ldots, n$

$$\mathbb{P}(|\xi_i| \geq \gamma) \geq \delta$$

then

$$K_p(X) \subset \frac{c(\delta)}{\gamma} K_p(\mathcal{E})$$
Stochastic domination and comparaison

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Moreover if $X$ satisfies the hypotheses of the main result then with proba $\geq 1 - 2 \exp(-C_1 N^{1-\alpha} n^\alpha)$, we have

$$\frac{\gamma}{c_2(\delta)} c_2 \left( B_\infty^n \cap \sqrt{\alpha \log \left( \frac{eN}{n} \right)} B_2^n \right) \subset \text{absconv}(X_1, \ldots, X_N)$$
Proof

Set $\Gamma = (X_1, \ldots, X_N)^*$ the matrix whose rows are $X_1, \ldots, X_N$. We need to prove that

$$\mathbb{P} \left( \inf_{t \in \partial K^p(X)} |\Gamma t|_\infty \geq 1/2 \right) \geq 1 - 2 \exp(-c_1 N^{1-\alpha} n^\alpha)$$

We define the set

$$\mathcal{F} = \{ f(\cdot) = 1_{|\langle \cdot, u \rangle| \geq 1/2}, \quad u \in \partial K_p \}$$

in such a way that

$$\frac{1}{N} \sum_{j=1}^{N} f(X_j) = \# \{ j, |\langle X_j, u \rangle| \geq 1/2 \}$$
Key tool - Concentration inequality

**Theorem (Talagrand 1996)**

Let $\mathcal{F}$ be a class of functions taking values in $\{0, 1\}$ such that $VC(\mathcal{F}) \leq d$ and $\sup_{f \in \mathcal{F}} E f^2 = \sigma^2$. The for every $x > 0$,

$$
\mathbb{P} \left( \sup_{f \in \mathcal{F}} \left| \frac{1}{N} \sum_{j=1}^{N} f(X_j) - E f \right| \geq R + x \right) \leq \exp \left( -N \frac{x^2/2}{\sigma^2 + 2R + x/3} \right)
$$

where $R \simeq \frac{d}{N} \log\left( \frac{c}{\sigma^2} \right) + \sigma \sqrt{\frac{d}{N} \log\left( \frac{c}{\sigma^2} \right)}$.

In our case, we have

$$
\mathcal{F} = \{ f(\cdot) = 1_{|\langle \cdot, u \rangle| \geq 1/2}, \quad u \in \partial K_p \}
$$

so that $VC(\mathcal{F}) \leq 10(n + 1)$