ON THE GAUSSIAN CONCENTRATION INEQUALITY AND ITS RELATION TO THE GAUSSIAN SURFACE AREA (PRELIMINARY VERSION)

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Abstract. Let γ_2 be a standard Gaussian measure in $\mathbb{R}^n$. For a given measurable set Q in $\mathbb{R}^n$, let H_Q be a half space in $\mathbb{R}^n$ such that $\gamma_2(Q) = \gamma_2(H_Q)$. The classical Gaussian concentration inequality states that for all measurable sets $Q \subset \mathbb{R}^n$ and for all $h > 0$,

$$\gamma_2(Q + hB_2^n) \geq \gamma_2(H_Q + hB_2^n).$$

Under some minor constrains on the set Q we obtain an improvement of the latter inequality in a certain range of h, depending on the Gaussian surface area of Q.

1. Introduction

We denote standard Gaussian measure on $\mathbb{R}^n$ by γ_2. For a measurable set $Q \subset \mathbb{R}^n$,

$$\gamma_2(Q) = \int_Q e^{-\frac{|y|^2}{2}} dy.$$

We recall that the Minkowski surface area of a convex set Q with respect to the Standard Gaussian measure is defined to be

$$\gamma_2(\partial Q) = \liminf_{\epsilon \to +0} \frac{\gamma_2((Q + \epsilon B_2^n) \setminus Q)}{\epsilon}, \tag{1}$$

where B_2^n denotes Euclidian ball in $\mathbb{R}^n$ and "+" stands for the Minkowski addition of sets.

Sudakov, Tsirelson [8] and Borell [4] proved, that among all convex sets of a fixed Gaussian measure, half spaces have the smallest Gaussian surface area. On the other hand, it was shown by Ball [1], that the Gaussian surface area of a convex set in $\mathbb{R}^n$ is asymptotically bounded from above by $Cn^{\frac{1}{4}}$, where C is an absolute constant. Nazarov [7]

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proved the sharpness of Ball's result:

$$0.28n^{\frac{1}{4}} \leq \max_{Q} \gamma_2(\partial Q) \leq 0.64n^{\frac{1}{4}}, \tag{2}$$

where the maximum is taken over all convex sets Q in $\mathbb{R}^n$.

Let (X, μ) be a compact metric space with a Borel probability measure μ. Let $B \subset X$ be a unit ball. The concentration function $\alpha(X, h)$ is defined to be

$$\alpha(X, h) = \sup_{\mu(A) \geq \frac{1}{2}} \left(1 - \mu(A + hB)\right),$$

where A is always a Borel subset of X (see [5], page 16).

Analogously, for a measurable set $Q \subset \mathbb{R}^n$ we define a function

$$\alpha_Q(h) : \mathbb{R}^+ \to \mathbb{R}$$

by

$$\alpha_Q(h) := 1 - \gamma_2(Q + hB_2^n).$$

It is well known (see, for example, [5], [2] or [3]) that for every measurable $Q \subset \mathbb{R}^n$ such that $\gamma_2(Q) \geq \frac{1}{2}$,

$$\alpha_Q(h) \leq \frac{1}{2} e^{-\frac{h^2}{2}}. \tag{3}$$

Moreover,

$$\gamma_2(Q + hB_2^n) \geq \gamma_2(H_Q + hB_2^n), \tag{4}$$

where H_Q is a half space such that $\gamma_2(Q) = \gamma_2(H_Q)$ (Theorem 1.2 in [5]).

In the present preprint we observe the relation of the estimates for $\alpha_Q(h)$ with the Gaussian surface area $\gamma_2(\partial Q)$. For some sets Q it allows us to improve the inequality (4) for certain range of h. Namely, we prove the following

Theorem 1.1. *For any convex set $Q \subset \mathbb{R}^n$ containing the origin and for any $0 \leq h \leq \frac{4\sqrt{n}}{\sqrt{\pi}\gamma_2(\partial Q)}$,*

$$\gamma_2(Q + hB_2^n) \geq \gamma_2(Q) + \frac{\sqrt{\pi}\gamma_2(\partial Q)^2}{8\sqrt{n}} \cdot \left(1 - e^{-\frac{\sqrt{n}}{\sqrt{\pi}\gamma_2(\partial Q)} h}\right). \tag{5}$$

Theorem 1.1 implies that for every convex set Q containing the origin, and for every $h > 0$,

$$\alpha_Q(h) \leq 1 - \gamma_2(Q) - \frac{\sqrt{\pi}\gamma_2(\partial Q)^2}{8\sqrt{n}} \cdot \left(1 - e^{-\frac{\sqrt{n}}{\sqrt{\pi}\gamma_2(\partial Q)} h}\right). \tag{6}$$

Let Q be a measurable set in $\mathbb{R}^n$ such that $\gamma_2(Q) = \gamma_2(H_r)$, where $H_r = \{x \in \mathbb{R}^n \,|\, \langle x, r\theta\rangle < 0\}$ for a unit vector θ. The classical concentration (4) implies that for every set Q, and for every $h > 0$,

$$\alpha_Q(h) \le 1 - \gamma_2(Q) - \frac{1}{\sqrt{2\pi}} \int_r^{r+h} e^{-\frac{t^2}{2}} dt. \tag{7}$$

Claim 1. *Let Q be a convex set in $\mathbb{R}^n$ such that $\gamma_2(Q) \ge \frac{1}{2}$ and $\gamma_2(\partial Q) \ge \frac{8}{\sqrt{2\pi}}$. Then the estimate (6) is stronger then (7) for all $h \in [0, c\frac{\gamma_2(\partial Q)\log\gamma_2(\partial Q)}{\sqrt{n}}]$, where c is an absolute constant.*

Proof. We observe that $r \ge 0$ since $\gamma_2(Q) \ge \frac{1}{2}$. Thus

$$\int_r^{r+h} e^{-\frac{t^2}{2}} dt \le \int_0^h e^{-\frac{t^2}{2}} dt.$$

Hence it suffices to show that

$$F(h) := \frac{\sqrt{\pi}\gamma_2(\partial Q)^2}{8\sqrt{n}} \cdot \left(1 - e^{-\frac{\sqrt{n}}{\sqrt{\pi}\gamma_2(\partial Q)}h}\right) - \frac{1}{\sqrt{2\pi}} \int_0^h e^{-\frac{t^2}{2}} dt \ge 0$$

on the interval $[0, c\frac{\gamma_2(\partial Q)\log\gamma_2(\partial Q)}{\sqrt{n}}]$. We find

$$F'(h) = \frac{\gamma_2(\partial Q)}{8} e^{-\frac{\sqrt{n}}{\sqrt{\pi}\gamma_2(\partial Q)}h} - \frac{1}{\sqrt{2\pi}} e^{-\frac{h^2}{2}}.$$

By taking the logarithm we obtain that $F'(h) \ge 0$ if and only if

$$h^2 - \frac{2\sqrt{n}}{\sqrt{\pi}\gamma_2(\partial Q)} h + 2\log\frac{\sqrt{2\pi}\gamma_2(\partial Q)}{8} \ge 0,$$

which happens in particular if $h \in [0, c\frac{\gamma_2(\partial Q)\log\gamma_2(\partial Q)}{\sqrt{n}}]$ (here we used the fact that $\gamma_2(\partial Q) \ge \frac{8}{\sqrt{2\pi}}$). We observe also that $F(0) = 0$. The function $F(h)$ is increasing on the interval $[0, c\frac{\gamma_2(\partial Q)\log\gamma_2(\partial Q)}{\sqrt{n}}]$, and thus positive. Which implies the Claim. □

2. Proof of Theorem 1.1

Let Q be a convex set in $\mathbb{R}^n$ containing the origin. It was shown in [7] (page 3) that

$$\gamma_2(\partial Q) \le \max_{y \in \partial Q} \frac{\sqrt{n}}{\sqrt{\pi}\langle y, n_y\rangle}, \tag{8}$$

where n_y stands for the normal vector at y.

The idea of the proof of the latter estimate is to consider a "polar coordinate system" associated with the body Q and to write

$$1=\gamma_2(\mathbb{R}^n)=\frac{1}{(\sqrt{2\pi})^n}\int_Q e^{-\frac{|y|^2}{2}}dy=$$

$$\frac{1}{(\sqrt{2\pi})^n}\int_{\partial Q}\int_0^{\infty} D(y,t)e^{-\frac{|X(y,t)|}{2}}dtd\sigma(y), \tag{9}$$

where $D(y,t)$ is the Jacobian of the change $(y,t)\to X(y,t)$. The inequality (8) follows when $X(y,t)=yt$.

Another estimate useful for the proof was shown in [6] (equation (78)) following the idea of [7]:

$$\gamma_2(Q+hB_2^n)\geq\gamma_2(Q)+\frac{1}{(\sqrt{2\pi})^n}\int_{\partial Q_2}\int_0^{h} e^{-\frac{|y+tn_y|^2}{2}}dtd\sigma(y). \tag{10}$$

The idea of the proof of the latter fact is to consider $X(y,t)=y+tn_y$ and apply the argument similar to (9).

We prove the following

Lemma 2.1. *Let Q be a convex set in $\mathbb{R}^n$. Let ρ be any positive number.*

(i) For any $h\in[0,2\rho]$,

$$\gamma_2(Q+hB_2^n)\geq\gamma_2(Q)+\left(\gamma_2(\partial Q)-\frac{\sqrt{n}}{\sqrt{\pi}\rho}\right)\cdot\frac{1}{2\rho}\cdot(1-e^{-2\rho h}). \tag{11}$$

(ii) For any $h\geq 2\rho$,

$$\gamma_2(Q+hB_2^n)\geq\gamma_2(Q)+\left(\gamma_2(\partial Q)-\frac{\sqrt{n}}{\sqrt{\pi}\rho}\right)\left(\frac{1}{2\rho}\cdot(1-e^{-4\rho^2})+(h-2\rho)\cdot e^{-h^2}\right).$$

Proof. Fix $\rho>0$. We split the surface of the body into two parts:

$$S_1=\{y\in\partial Q:\ \langle y,n_y\rangle\geq\rho\}$$

and

$$S_2=\{y\in\partial Q:\ \langle y,n_y\rangle<\rho\}.$$

By (8), $\gamma_2(S_1)\leq\frac{\sqrt{n}}{\sqrt{\pi}\rho}$. Thus,

$$\gamma_2(S_2)\geq\gamma_2(\partial Q)-\frac{\sqrt{n}}{\sqrt{\pi}\rho}. \tag{12}$$

The inequality (10) entails that

$$\gamma_2(Q+hB_2^n)\geq\gamma_2(Q)+\frac{1}{(\sqrt{2\pi})^n}\int_{S_2}\int_0^{h} e^{-\frac{|y+tn_y|^2}{2}}dtd\sigma(y),$$

since $S_2\subset\partial Q$.

We observe, that for any $y \in S_2$,

$$|y + tn_y|^2 = |y|^2 + t^2 + 2t\langle y, n_y\rangle \le |y|^2 + t^2 + 2t\rho.$$

Thus

$$\gamma_2(Q + hB_2^n) \ge \gamma_2(Q) + \frac{1}{(\sqrt{2\pi})^n}\int_{S_2}\int_0^h e^{-\frac{|y|^2+t^2+2t\rho}{2}}dtd\sigma(y) =$$

$$\gamma_2(Q) + \gamma_2(S_2)\cdot\int_0^h e^{-\frac{t^2+2t\rho}{2}}dt.$$

Using (12), we obtain that

$$\gamma_2(Q + hB_2^n) \ge \gamma_2(Q) + (\gamma_2(\partial Q) - \frac{\sqrt{n}}{\sqrt{\pi}\rho})\cdot\int_0^h e^{-\frac{t^2+2t\rho}{2}}dt. \tag{13}$$

If $h \le 2\rho$, then $t^2 \le 2t\rho$ for every $t \in [0, h]$. Thus in this case

$$\int_0^h e^{-\frac{t^2+2t\rho}{2}}dt \ge \int_0^h e^{-2t\rho}dt = \frac{1}{2\rho}\cdot(1 - e^{-2\rho h}). \tag{14}$$

For $h \ge 2\rho$ we estimate

$$\int_0^h e^{-\frac{t^2+2t\rho}{2}}dt \ge \int_0^{2\rho} e^{-2t\rho}dt + \int_{2\rho}^h e^{-t^2}dt \ge$$

$$\frac{1}{2\rho}\cdot(1 - e^{-4\rho^2}) + (h - 2\rho)e^{-h^2}. \tag{15}$$

Gluing (13) and (14) together we obtain the first part of the Lemma; the second part follows from (13) and (15). □

To finish the proof of Theorem 1.1 we plug $\rho = \frac{2\sqrt{n}}{\sqrt{\pi}\gamma_2(\partial Q)}$ into (11). □

Theorem 1.1 is one of the possible corollaries of Lemma 2.1 which illustrates the use of the estimate; however, Lemma 2.1 may be of separate interest and imply other estimates which may be better in different ranges of h.

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