MAXIMAL SURFACE AREA OF POLYTOPES WITH RESPECT TO LOG-CONCAVE ROTATION INVARIANT MEASURES.

GALYNA LIVSHYTS

Abstract. It was shown in [21] that the maximal surface area of a convex set in $\mathbb{R}^n$ with respect to a rotation invariant log-concave probability measure γ is of order $\frac{\sqrt{n}}{\sqrt[4]{Var|X|}\sqrt{\mathbb{E}|X|}}$, where X is a random vector in $\mathbb{R}^n$ distributed with respect to γ. In the present paper we discuss surface area of convex polytopes P_K with K facets. We find tight bounds on the maximal surface area of P_K in terms of K. We show that $\gamma(\partial P_K) \lesssim \frac{\sqrt{n}}{\mathbb{E}|X|} \cdot \sqrt{\log K} \cdot \log n$ for all K. This bound is better then the general bound for all $K \in [2, e^{\frac{c}{\sqrt{Var|X|}}}]$. Moreover, for all K in that range the bound is exact up to a factor of $\log n$: for each $K \in [2, e^{\frac{c}{\sqrt{Var|X|}}}]$ there exists a polytope P_K with at most K facets such that $\gamma(\partial P_K) \gtrsim \frac{\sqrt{n}}{\mathbb{E}|X|}\sqrt{\log K}$.

1. Introduction

In this paper we study properties of the surface area of convex polytopes with respect to log-concave rotation invariant probability measures. For sets $A, B \subset \mathbb{R}^n$ the Minkowski sum is defined as

$$A + B = \{a + b \,|\, a \in A, b \in B\}.$$

For a scalar λ the dilated set is

$$\lambda A := \{\lambda a \,|\, a \in A\}.$$

A measure γ on $\mathbb{R}^n$ is called log-concave if for any measurable sets $A, B \subset \mathbb{R}^n$ and for any $\lambda \in [0, 1]$,

$$\gamma(\lambda A + (1-\lambda)B) \geq \gamma(A)^{\lambda}\gamma(B)^{1-\lambda}.$$

It was shown by Borrell [6], that a measure is log-concave if and only if it has a density with respect to the Lebesgue measure on some affine

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hyperplane, and this density is a log-concave function. Log-concave measures have been studied intensively in the recent years. For the background and numerous interesting properties, see for example [16], [17], [19] and [23].

A measure γ is called rotation invariant if, for every rotation T and for every measurable set A,

$$\gamma(TA) = \gamma(A).$$

Log-concave rotation invariant measures appear for example in [19], [2], [3], [4] and [21].

In the present paper we restrict our attention to probability measures (that means that the measure of the whole space is equal to 1). Examples of log-concave rotation invariant probability measures are the Standard Gaussian Measure γ_2 and the Lebesgue measure restricted on a ball.

Let X be a random vector in $\mathbb{R}^n$ distributed with respect to a measure γ. We introduce

$$\mathbf{E} := \mathbb{E}|X| \tag{1}$$

and

$$\mathbf{S} := \frac{\sqrt{\mathbb{E}\left(|X| - \mathbb{E}|X|\right)^2}}{\mathbb{E}|X|}, \tag{2}$$

the expectation and the normalized standard deviation of the absolute value of X. $\mathbf{E}$ and $\mathbf{S}$ are natural parameters of the measure γ. For rotation invariant measures $\mathbf{S} \in [\frac{c_1}{n}, \frac{c_2}{\sqrt{n}}]$, where c and c' are absolute constants (see [16] or [21], Remark 2.9). The parameter $\mathbf{S}$ is closely related to $\sigma = \sqrt{\mathbb{E}\left(|X| - \mathbb{E}|X|\right)^2}$. The famous Thin Shell Conjecture suggests that σ is bounded from above by an absolute constant for all isotropic (see, for example, [23], [16] for definitions and properties) log-concave measures (see [16], [17], [18], [11], [13], [10], [13]). Currently, the best bound is $Cn^{\frac{1}{3}}$ and is due to Gudeon and E. Milman [13].

The Minkowski surface area of a convex set Q with respect to the measure γ is defined to be

$$\gamma(\partial Q) = \liminf_{\epsilon \to +0} \frac{\gamma((Q + \epsilon B_2^n) \setminus Q)}{\epsilon}, \tag{3}$$

where B_2^n denotes Euclidian ball in $\mathbb{R}^n$. In many cases the Minkowski surface area has an integral representation:

$$\gamma(\partial Q) = \int_{\partial Q} f(y) d\sigma(y), \tag{4}$$

where $f(y)$ is the density γ and $d\sigma(y)$ stands for the Lebesgue surface measure on ∂Q (see for example Appendix of [14] and Appendix of [21]).

The questions of estimating the surface area of $n-$dimensional convex sets with respect to the Standard Gaussian Measure have been actively studied. Sudakov, Tsirelson [26] and Borell [5] proved, that among all convex sets of a fixed Gaussian volume, half spaces have the smallest Gaussian surface area. Mushtari and Kwapien asked the reverse version of the isoperimetric inequality, i.e. how large the Gaussian surface area of a convex set $Q \subset \mathbb{R}^n$ can be. It was shown by Ball [1], that Gaussian surface area of a convex set in $\mathbb{R}^n$ is asymptotically bounded by $Cn^{\frac{1}{4}}$, where C is an absolute constant. Nazarov [24] proved the sharpness of Ball's result and gave the complete solution to this asymptotic problem:

$$0.28n^{\frac{1}{4}} \leq \max_{Q \in \mathcal{K}_n} \gamma_2(\partial Q) \leq 0.64n^{\frac{1}{4}}, \tag{5}$$

where by $\mathcal{K}_n$ we denote the set of all convex sets in $\mathbb{R}^n$.

Further estimates for $\gamma_2(\partial Q)$ for the special case of polynomial level set surfaces were provided by Kane [14]. He showed that for any polynomial $P(y)$ of degree d, $\gamma_2(P(y)=0) \leq \frac{d}{\sqrt{2}}$.

For the case of all rotation invariant log-concave measures it was shown in [21], that

$$\max_{Q \in \mathcal{K}_n} \gamma(\partial Q) \approx \frac{\sqrt{n}}{\mathbf{E} \cdot \sqrt{\mathbf{S}}}.$$

Let K be a given positive integer. In the present paper we consider the family of $n-$dimensional convex polytopes with K facets, where by a “polytope” we mean the intersection of K half-spaces (we do not assume compactness as it is irrelevant for the type of questions we consider). We obtain the bounds on the surface area of the polytope with K facets with respect to rotation invariant log-concave measure γ in terms of K and the natural parameters of γ.

It is not hard to show that the $\gamma-$surface area of any half-space does not exceed $C\frac{\sqrt{n}}{\mathbf{E}}$, for some absolute constant C (see (21) below). Thus the immediate bound for the surface area of a polytope with K facets is $C\frac{K\sqrt{n}}{\mathbf{E}}$. In the present paper we show a sharper estimate from above. We also show an estimate from below on the maximal surface area of a convex polytope with K facets. Both of the estimates match up to a $\log n$ factor.

The estimate from above is the content of the following Theorem:

Theorem 1.1. *Let $n \geq 2$. Fix positive integer $K \in [2, e^{\frac{c}{\boldsymbol{S}}}]$. Let P be a convex polytope in $\mathbb{R}^n$ with at most K facets. Let γ be a rotation invariant log-concave measure with $\boldsymbol{E}$ and $\boldsymbol{S}$ defined by (1) and (2). Then*

$$\gamma(\partial P) \leq C\frac{\sqrt{n}}{\boldsymbol{E}} \cdot \sqrt{\log K} \cdot \log \frac{1}{\boldsymbol{S} \log K},$$

where C and c stand for absolute constants.

Remark 1.2. *We note that for $K \geq e^{\frac{c}{\boldsymbol{S}}}$ the bound from Theorem 1.1 becomes worse then the general bound $\frac{\sqrt{n}}{\sqrt[4]{Var|X|}\sqrt{\mathbb{E}|X|}}$. The latter bound is also optimal for all $K \geq e^{\frac{c}{\boldsymbol{S}}}$.*

Remark 1.3. *Since $\boldsymbol{S} \in [\frac{c}{n}, \frac{C}{\sqrt{n}}]$ for all log-concave rotation invariant measures, Theorem 1.1 reads in fact, that for all $K \in [1, e^{\frac{c}{\boldsymbol{S}}}]$,*

$$\gamma(\partial P) \leq C\frac{\sqrt{n}}{\boldsymbol{E}} \cdot \sqrt{\log K} \cdot \log \frac{n}{\log K} \leq C\frac{\sqrt{n}}{\boldsymbol{E}} \cdot \sqrt{\log K} \cdot \log n.$$

Theorem 1.1, up to a log factor, is a generalization of the following Theorem of Nazarov [25]:

Theorem 1.4 (F. Nazarov). *Let $n \geq 2$ and $K \geq 2$ be integers. Let P be a convex polytope in $\mathbb{R}^n$ with at most K facets. Let γ_2 be the Standard Gaussian Measure. Then there exist a positive constant C such that*

$$\gamma_2(\partial P) \leq C\sqrt{\log K}.$$

For a generalization of the Theorem of Nazarov in an entirely different set up see [15]. See also Section 5 of the present paper for the proof of the analogous result for measures with densities $C(n,p)e^{-\frac{|y|^p}{p}}$. The case $p = 2$ corresponds to the Gaussian measure. Theorem 5.1 from Section 5 is a generalization of the Theorem of Nazarov.

We also obtain a lower bound for the maximal surface area of a convex polytope with K facets. It proves sharpness of Theorem 1.4 of Nazarov. It also shows sharpness of Theorem 1.1 up to a $\log n$ factor:

Theorem 1.5. *Let $n \geq 2$. Let γ be a rotation invariant log-concave measure with $\boldsymbol{E}$ and $\boldsymbol{S}$ defined by (1) and (2). Fix positive integer $K \in [2, e^{\frac{c}{\boldsymbol{S}}}]$. Then there exists a convex polytope P in $\mathbb{R}^n$ with at most K facets such that*

$$\gamma(\partial P) \geq C'\frac{\sqrt{n}}{\boldsymbol{E}}\sqrt{\log K},$$

where c and C' stand for absolute constants.

The next section is dedicated to some technical preliminaries. In Section 3 we give the proof of Theorem 1.1. In Section 4 we prove Theorem 1.5. Finally, in Section 5 we show that Theorem 1.1 can be refined in some partial cases of measures, which include the Standard Gaussian measure.

2. Preliminaries and definitions

This section is dedicated to some general properties of rotation invariant log-concave measures. We outline some elementary facts which are needed for the proof. Some of them have appeared in literature. See [16] for an excellent overview of the properties of log-concave measures; see also [21] for more details and the proofs of the facts listed in the present section.

We use notation $|\cdot|$ for the norm in Euclidean space $\mathbb{R}^n$; $|A|$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$. We write $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ for the unit ball in $\mathbb{R}^n$ and $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ for the unit sphere. We denote $\nu_n = |B_2^n| = \frac{\pi^{\frac{n}{2}}}{\Gamma(n/2+1)}$.

We shall use notation $\lesssim$ for an asymptotic inequality: we say that $A(n) \lesssim B(n)$ if there exists an absolute positive constant C (independent of n), such that $A(n) \leq C \cdot B(n)$. Correspondingly, $A(n) \approx B(n)$ means that $B(n) \lesssim A(n) \lesssim B(n)$. Also in the present paper C, c, c_1 etc denote absolute constants which may change from line to line.

We fix a convex nondecreasing function $\varphi(t) : [0, \infty) \to [0, \infty]$. Let γ be a probability measure on $\mathbb{R}^n$ with density $C_n e^{-\varphi(|y|)}$. The normalizing constant C_n equals to $[n\nu_n J_{n-1}]^{-1}$, where

$$J_{n-1} = \int_0^\infty t^{n-1} e^{-\varphi(|y|)} dt. \tag{6}$$

The measure γ is rotation invariant and log-concave; conversely, every rotation invariant log-concave measure is representable this way in terms of some convex function φ. Since we normalize the measure anyway, we may assume that $\varphi(0) = 0$. We will be aiming for the estimates for

$$\gamma(\partial P) = [n\nu_n J_{n-1}]^{-1} \int_{\partial P} e^{-\varphi(|y|)} d\sigma(y),$$

where P is a convex polytope with K facets.

Without loss of generality we assume that $\varphi \in C^2[0, \infty)$. This can be shown by the standard smoothing argument (see, for example, [9]).

We introduce the notation

$$g_{n-1}(t) = t^{n-1} e^{-\varphi(t)}. \tag{7}$$

Definition 2.1. *We define t_0 to be the point of maxima of the function $g_{n-1}(t)$, i.e., t_0 is the solution of the equation*

$$\varphi'(t)t = n-1. \tag{8}$$

The equation (8) has a solution, since $t\varphi'(t)$ is non-decreasing, continuous and $\lim_{t\to+\infty} t\varphi'(t) = +\infty$. This solution is unique, since $t\varphi'(t)$ strictly increases on its support. This definition appears in most of the literature dedicated to spherically symmetric log-concave measures: see, for example, [19] or [16] (Lemma 4.3), as well as [21] (Definition 2).

The following Lemma is proved in [19]. It provides asymptotic bounds for J_{n-1}.

Lemma 2.2.

$$\frac{g_{n-1}(t_0)t_0}{n} \leq J_{n-1} \leq \sqrt{2\pi}(1+o(1))\frac{g_{n-1}(t_0)t_0}{\sqrt{n-1}}.$$

The following definitions appear in [21] (Definition 3).

Definition 2.3. *Define the "outer" λ_o to be a positive number satisfying:*

$$\varphi(t_0(1+\lambda_o)) - \varphi(t_0) - (n-1)\log(1+\lambda_o) = 1. \tag{9}$$

Similarly, define the "inner" λ_i as follows:

$$\varphi(t_0(1-\lambda_i)) - \varphi(t_0) - (n-1)\log(1-\lambda_i) = 1. \tag{10}$$

We put

$$\lambda := \lambda_i + \lambda_o. \tag{11}$$

We note that (9) is equivalent to

$$g_{n-1}(t_0) = e \cdot g_{n-1}(t_0(1+\lambda_o)), \tag{12}$$

and (10) is equivalent to

$$g_{n-1}(t_0) = e \cdot g_{n-1}(t_0(1-\lambda_i)). \tag{13}$$

Parameter λ from (11) has a nice property.

Lemma 2.4.

$$J_{n-1} \approx \lambda t_0 g_{n-1}(t_0).$$

See [21] (Lemma 4) for the details and the proof. The following fact is also presented in [21] (Lemma 5).

Lemma 2.5. *For all $n \geq 2$,*

$$\frac{J_n}{J_{n-1}} \approx t_0.$$

The above implies, that $t_0 \approx \mathbf{E} = \mathbb{E}|X|$, where X is a random vector in $\mathbb{R}^n$ distributed with respect to γ. Also, $\lambda \approx \mathbf{S}$, where $\mathbf{S}$ is defined by (2) (see [21] (Lemmas 9 and 10) for the details).

Remark 2.6. *We note that Lemma 2.4 together with Lemma 2.2 imply that* $\lambda \in [\frac{c_1'}{n}, \frac{c_2'}{\sqrt{n}}]$. *Both of the estimates are exact: it is equal to* $\frac{c}{n}$ *for Lebesgue measure concentrated on a ball and to* $\frac{C}{\sqrt{n}}$ *for Standard Gaussian Measure.*

Since $\lambda \approx \boldsymbol{S}$, *we claim also that* $\boldsymbol{S} \in [\frac{c_1}{n}, \frac{c_2}{\sqrt{n}}]$.

We are now after the restated versions of the Theorems 1.1 and 1.5:

Theorem 2.7. *Let* $n \geq 2$. *Let* $K \in [2, e^{\frac{c}{\lambda}}]$. *Let* P *be a convex polytope in* $\mathbb{R}^n$ *with at most* K *facets. Let* γ *be a rotation invariant log-concave measure. Then*

$$\gamma(\partial P) \leq C \frac{\sqrt{n}}{t_0} \cdot \sqrt{\log K} \cdot \log \frac{1}{\lambda \log K}.$$

and

Theorem 2.8. *Let* $n \geq 2$. *Let* γ *be a rotation invariant log-concave measure. Fix positive integer* $K \in [2, e^{\frac{c}{\lambda}}]$. *Then there exists a polytope* P *in* $\mathbb{R}^n$ *with at most* K *facets such that*

$$\gamma(\partial P) \geq C' \frac{\sqrt{n}}{t_0} \sqrt{\log K}.$$

The following Lemma is an elementary fact about log-concave functions (for example, it appears in [21] as Lemma 3).

Lemma 2.9. *Let* $g(t) = e^{f(t)}$ *be a log-concave function on [a,b] (where both a and b may be infinite). We assume that* $f \in C^2[a, b]$. *Let* t_0 *be the point of maxima of* $f(t)$. *Assume that* $t_0 > 0$. *Consider* $x > 0$ *and* $\psi > 0$ *such that*

$$f(t_0) - f((1 + x)t_0) \geq \psi.$$

Then,

$$\int_{(1+x)t_0}^{b} g(t)dt \leq \frac{x t_0 g(t_0)}{\psi e^{\psi}}.$$

Similarly, if $f(t_0) - f((1 - x)t_0) \geq \psi$,

$$\int_{a}^{(1-x)t_0} g(t)dt \leq \frac{x t_0 g(t_0)}{\psi e^{\psi}}.$$

The next Lemma is similar to Lemma 12 from [21].

Lemma 2.10. *Pick $\psi \in [1, c\log\frac{1}{\lambda}]$. Define μ to be smallest positive number such that*

$$\varphi\left(t_0(1+\mu)\right) - \varphi(t_0) - (n-1)\log(1+\mu) \geq \psi. \tag{14}$$

Define

$$A := (1+\mu)t_0 B_2^n \setminus \frac{t_0}{2e} B_2^n.$$

We claim, that such μ is well-defined and

$$\gamma(\partial Q \setminus A) \precsim \frac{\sqrt{n}}{t_0 \lambda \sqrt{\psi} e^{\psi}}.$$

Proof. First, consider $M = Q \cap \frac{t_0}{2e} B_2^{n+1}$. Then,

$$\gamma(M) \leq \frac{1}{(n-1)\nu_n J_{n-1}} \int_M e^{-\varphi(|y|)} d\sigma(y) \leq \frac{|M|}{(n-1)\nu_n J_{n-1}} \leq$$

$$\frac{|\frac{t_0}{2e}\mathbb{S}^{n-1}|}{(n-1)\nu_n J_{n-1}} \approx \frac{t_0^{n-1}}{(2e)^{n-1}\lambda t_0 e^{-\varphi(t_0)} t_0^{n-1}} = \frac{1}{\lambda t_0} \cdot \frac{e^{\varphi(t_0)}}{(2e)^{n-1}}, \tag{15}$$

where the equivalency follows from Lemma 2.4 and (18). By the Mean Value Theorem, $\varphi(t_0) \leq n$, so we estimate (15) from above by $c\frac{2^{-n}}{\lambda t_0}$. In a view of Remark 2.6, the latter bound is much better then the one stated in the Lemma.

Next, let $N = \partial Q \setminus (1+\mu)t_0 B_2^n$. In the current range of ψ, we observe:

$$\psi = \frac{(n-1)\mu^2}{2} + o(1). \tag{16}$$

We obtain the following integral expression for $e^{-\varphi(|y|)}$ (inspired by [1]):

$$e^{-\varphi(|y|)} = \int_{|y|}^{\infty} \varphi'(t) e^{-\varphi(t)} dt = \int_0^{\infty} \varphi'(t) e^{-\varphi(t)} \chi_{[0,t]}(|y|) dt,$$

where $\chi_{[0,t]}$ stands for characteristic function of the interval $[0,t]$. In the current range of $|y|$,

$$e^{-\varphi(|y|)} = \int_{(1+\mu)t_0}^{\infty} \varphi'(t) e^{-\varphi(t)} \chi_{[0,t]}(|y|) dt.$$

Using the above, passing to the polar coordinates and integrating by parts, we get

$$\gamma(N) \leq \frac{1}{J_{n-1}} \int_{(1+\mu)t_0}^{\infty} t^{n-1} \varphi'(t) e^{-\varphi(t)} dt \approx$$

$$\frac{g_{n-1}((1+\mu)t_0) + (n-1)\int_{(1+\mu)t_0}^{\infty} g_{n-2}(t) dt}{\lambda t_0 g_{n-1}(t_0)}. \tag{17}$$

Lemma 2.9, applied with $x = \mu$ and ψ, together with (16) entails that (17) is asymptotically less than

$$\frac{e^{-\psi}}{\lambda t_0} + \frac{n\mu}{\lambda t_0 \psi e^{\psi}} \leq \frac{1}{\lambda t_0} \cdot (1 + \frac{\mu n}{\psi}) e^{-\psi} \precsim \frac{\sqrt{n}}{t_0 \lambda \sqrt{\psi} e^{\psi}},$$

which implies the estimate. □

We use Lemma 2.10 with $\mu \approx \frac{\sqrt{\log \frac{1}{\lambda \sqrt{\log K}}}}{\sqrt{n}} \leq 1$. We get, that for

$$A := \left(1 + \frac{\sqrt{\log \frac{1}{\lambda \sqrt{\log K}}}}{\sqrt{n}}\right) t_0 B_2^n \setminus \frac{t_0}{2e} B_2^n,$$

it holds that

$$\gamma(\partial Q \setminus A) \precsim \frac{\sqrt{n}}{t_0} \sqrt{\log K}.$$

Let $y \in \partial Q$. In an account of the above, we may assume that

$$|y| \approx t_0 \tag{18}$$

throughout the proof.

We consider the hyperplane H passing through the origin.

$$\gamma(H) \approx \frac{1}{n \nu_n J_{n-1}} \int_{\mathbb{R}^{n-1}} e^{-\varphi(|y|)} d\sigma(y) = \frac{(n-1)\nu_{n-1} J_{n-2}}{n \nu_n J_{n-1}}. \tag{19}$$

It is well known that

$$\frac{\nu_{n-1}}{\nu_n} \approx \sqrt{n}. \tag{20}$$

Applying (20) together with Lemma 2.5 and (19), we obtain that

$$\gamma(H) \approx \frac{\sqrt{n}}{t_0}. \tag{21}$$

Thus the trivial bound on the surface area of a polytope P with K facets in $\mathbb{R}^n$ is $\frac{\sqrt{n}}{t_0} K = \frac{\sqrt{n}}{\mathbf{E}} K$. We shall improve it.

3. Proof of the upper bound part

Let Q be a convex set in $\mathbb{R}^n$. For $y \in \partial Q$ define

$$\alpha(y) := \cos(y, n_y), \tag{22}$$

where n_y stands for the normal vector at y. We also define

$$\psi(y) := \log \frac{g_{n-1}(t_0)}{g_{n-1}(|y|)} = \varphi(t_0) - \varphi(|y|) - (n-1) \log \frac{|y|}{t_0}. \tag{23}$$

It was shown in [21] (Equation (46)) that

$$\gamma(\partial Q) \lesssim \max_{y \in \partial Q} \frac{1}{\lambda |y| \alpha(y) e^{\psi(y)}}. \tag{24}$$

It was also shown in [21] (Equation (49) and Proposition 1) that

$$\gamma(\partial Q) \lesssim \max_{y \in \partial Q} \sqrt{n} \sqrt{\psi(y)} \frac{\alpha(y)\sqrt{n}\sqrt{\psi(y)} + 1}{|y|}. \tag{25}$$

Pick any $\theta \in \mathbb{S}^{n-1}$ and $\rho > 0$. Let

$$H_\rho = \{x \in \mathbb{R}^n \mid \langle x, \theta \rangle = \rho\}$$

be a hyperplane at distance ρ from the origin. We note that for $y \in H_\rho$, $\alpha(y)|y| = \rho$. So we introduce another function

$$r(y) := \frac{\sqrt{n}}{t_0} \alpha(y)|y|. \tag{26}$$

For all $y \in H_\rho$ the function $r(y) = \frac{\sqrt{n}}{t_0}\rho$. Applying (18), we rewrite (24) and (25) in terms of $r(y)$:

$$\gamma(\partial Q) \lesssim \frac{\sqrt{n}}{t_0} \max_{y \in \partial Q} \frac{1}{\lambda r(y) e^{\psi(y)}}, \tag{27}$$

$$\gamma(\partial Q) \lesssim \frac{\sqrt{n}}{t_0} \max_{y \in \partial Q} \left(r(y)\psi(y) + \sqrt{\psi(y)} \right). \tag{28}$$

We are going to estimate the measure of each facet using both (27) and (28), and "the breaking point" is going to depend on how far the facet is from the origin. So we minimize the expression

$$\frac{1}{\lambda r e^{\psi}} + r\psi + \sqrt{\psi} \tag{29}$$

in ψ in terms of r. If $r \lesssim \frac{1}{\sqrt{\psi}}$, the minimum of (29) is equivalent to the minimum of $\frac{1}{\lambda r e^{\psi}} + \sqrt{\psi}$ which is achieved when $\psi \approx \log \frac{1}{\lambda r}$ and is approximately equal to $\sqrt{\log \frac{1}{\lambda r}}$. If $r \gtrsim \frac{1}{\sqrt{\psi}}$, the minimum of (29) is equivalent to the minimum of $\frac{1}{\lambda r e^{\psi}} + r\psi$ which is achieved when $\psi \approx \log \frac{1}{r^2 \lambda}$ and is approximately equal to $r \log \frac{1}{r^2\lambda}$. We conclude that the minimum of (29) is asymptotically less then

$$\max\left(\sqrt{\log \frac{1}{\lambda r}}, r \log \frac{1}{\lambda r^2} \right).$$

We fix a positive number R (which we will select later). Consider a convex polytope P_1 such that all its facets are close enough to the origin. In other words, assume that $r(y) < R$ for all $y \in \partial P_1$. Then

$$\gamma(\partial P_1) \lesssim \frac{\sqrt{n}}{t_0} \max_{r\in[0,R]} \left(\sqrt{\log \frac{1}{\lambda r}}, r \log \frac{1}{\lambda r^2} \right). \tag{30}$$

We note that for $R \in (0,1)$, the right hand side of (30) is infinitely large. But as long as we assume that $R \in (1, \frac{1}{e\sqrt{\lambda}})$ the right hand side of (30) is asymptotically equal to

$$\frac{\sqrt{n}}{t_0} \max_{r\in[0,R]} \left(r \log \frac{1}{\lambda r^2} \right) \approx \frac{\sqrt{n}}{t_0} R \log \frac{1}{\lambda R^2}, \tag{31}$$

since $r \log \frac{1}{\lambda r^2}$ is increasing on $(1, \frac{1}{e\sqrt{\lambda}})$.

The estimate (31) is the first key ingredient for our proof. The other key ingredient is the following Lemma.

Lemma 3.1.
$$\gamma(H_\rho) \lesssim \frac{\sqrt{n}}{t_0} \left(e^{-n} + e^{-c\frac{n\rho^2}{t_0^2}} \right),$$
where c is an absolute constant.

Proof. We write
$$\gamma(H_\rho) = \frac{1}{n\nu_n J_{n-1}} \int_{\mathbb{R}^{n-1}} e^{-\varphi(\sqrt{|y|^2+\rho^2})} d\sigma(y).$$
Passing to the polar coordinates in $\mathbb{R}^{n-1}$, we get:
$$\gamma(H_\rho) = \frac{(n-1)\nu_{n-1}}{n\nu_n J_{n-1}} \int_0^\infty s^{n-2} e^{-\varphi(\sqrt{s^2+\rho^2})} ds.$$
We make a change of variables $t = \sqrt{s^2+\rho^2}$ and use (20):
$$\gamma(H_\rho) = \frac{(n-1)\nu_{n-1}}{n\nu_n J_{n-1}} \int_\rho^\infty t^{n-2} (1 - \frac{\rho^2}{t^2})^{\frac{n-2}{2}} e^{-\varphi(t)} \frac{t}{\sqrt{t^2-\rho^2}} dt \approx$$

$$\frac{\sqrt{n}}{J_{n-1}} \int_\rho^\infty t^{n-2} (1 - \frac{\rho^2}{t^2})^{\frac{n-3}{2}} e^{-\varphi(t)} dt. \tag{32}$$

It was shown in [19] (Lemma 2.1) that

$$\int_{5t_0}^\infty t^{n-2} e^{-\varphi(t)} dt \le e^{-n} J_{n-2}. \tag{33}$$

We note that $(1-\frac{\rho^2}{t^2})^{\frac{n-3}{2}} \leq 1$ for $n \geq 3$. Applying (33) together with Lemma 2.5, we conclude that for $n \geq 3$, (32) is asymptotically smaller then

$$\frac{\sqrt{n}}{t_0}\left(e^{-n} + \max_{t\in[\rho,5t_0]}(1-\frac{\rho^2}{t^2})^{\frac{n-2}{2}}\right) \lesssim \frac{\sqrt{n}}{t_0}\left(e^{-n} + e^{-c\frac{n\rho^2}{t_0^2}}\right).$$

For $n = 2$ the surface area of any convex set is bounded by a constant. Thus for $n = 2$ the result follows with the proper choice of C in Theorem 1.1. This concludes the proof of the Lemma. □

Consider a polytope P_2 with K facets such that all its facets are far enough from the origin. Namely, assume that $r(y) \geq R$ for all $y \in \partial P_2$. Then Lemma 3.1 implies that

$$\gamma(\partial P_2) \lesssim \frac{\sqrt{n}}{t_0} K e^{-cR^2}, \tag{34}$$

as long as we chose $R \lesssim \sqrt{n}$.

Now we glue everything together. Let $R \in (1, \frac{1}{e\sqrt{\lambda}})$ (note that (34) is applicable for this range of R since by Remark 2.3, $\frac{1}{e\sqrt{\lambda}} \lesssim \sqrt{n}$). We split the surface of our polytope P into two parts P_1 and P_2, where P_1 consists of the facets which are closer then R to the origin and P_2 is the rest, i.e. the facets which are farther then R from the origin. In other words,

$$P_1 = \{y \in \partial P \,|\, r(y) \leq R\}$$

and

$$P_2 = \{y \in \partial P \,|\, r(y) > R\}.$$

Applying (31) and (34) we observe, that

$$\gamma(\partial P) \lesssim \frac{\sqrt{n}}{t_0}\left(R \log\frac{1}{\lambda R^2} + K e^{-R^2}\right). \tag{35}$$

The estimate (35) holds for every $R \in (1, \frac{1}{e\sqrt{\lambda}})$. Minimizing (35) in R we get that

$$\gamma(\partial P) \lesssim \frac{\sqrt{n}}{t_0}\sqrt{\log K}\log\frac{1}{\lambda \log K}. \tag{36}$$

Here we plugged $R \approx \sqrt{\log K}$, so the above estimate is valid for all $K \in [1, e^{\frac{c}{\lambda}}]$ for some absolute constant c. This finishes the proof of Theorem 2.7, and thus Theorem 1.1.□

4. Proof of the lower bound part

Fix an integer $K \le e^{\frac{c}{\lambda}}$. We consider K independent uniformly distributed random vectors $x_i \in \mathbb{S}^{n-1}$. Let $\rho \in (0, c\frac{t_0}{\sqrt{\lambda n}})$ (we will chose it later). Consider a random polytope P in $\mathbb{R}^n$, circumscribed around the ball of radius ρ:

$$P = \{x \in \mathbb{R}^n : \langle x, x_i \rangle \le \rho, \ \forall i = 1, ..., K\}.$$

Passing to the polar coordinates as in Lemma 3.1 and restricting the integration to $[t_0(1-\lambda), t_0(1+\lambda)]$ we estimate the expectation of $\gamma(\partial P)$ from below:

$$\mathbb{E}(\gamma(\partial P)) \gtrsim$$

$$\frac{1}{n\nu_n J_{n-1}} K(n-1)\nu_{n-1} \int_{t_0(1-\lambda)}^{t_0(1+\lambda)} t^{n-2} e^{-\varphi(t)} (1 - \frac{\rho^2}{t_0^2})^{\frac{n-3}{2}} (1-p(t))^{K-1} dt \gtrsim$$

$$\frac{\sqrt{n}}{J_{n-1}} K \left(1 - \frac{\rho^2}{t_0^2(1-\lambda)^2}\right)^{\frac{n-3}{2}} \int_{t_0(1-\lambda)}^{t_0(1+\lambda)} e^{-\varphi(t)} t^{n-2} (1-p(t))^{K-1} dt, \tag{37}$$

where $p(t)$ is the probability that the fixed point on the sphere of radius t is separated from the origin by the hyperplane $H_i = \{x : \langle x, x_i \rangle = \rho\}$. It was shown in [21], Equation (70) (see also [24]), that for $t \in [(1-\lambda)t_0, (1+\lambda)t_0]$

$$p(t) \lesssim \frac{t_0}{\sqrt{n}\rho} \left(1 - \frac{\rho^2}{t_0^2(1+\lambda)^2}\right)^{\frac{n-3}{2}}. \tag{38}$$

We chose ρ so that

$$K^{-1} = \frac{t_0}{\sqrt{n}\rho} \left(1 - \frac{\rho^2}{t_0^2(1+\lambda)^2}\right)^{\frac{n-3}{2}},$$

which in the current range of ρ means that $\rho = c\frac{t_0}{\sqrt{n}}\sqrt{\log K}$, and

$$K \left(1 - \frac{\rho^2}{t_0^2(1-\lambda)^2}\right)^{\frac{n-3}{2}} = \frac{\sqrt{n}\rho}{t_0}.$$

We use the above together with (37) to conclude that the expectation $\mathbb{E}(\gamma(\partial P))$ is greater than

$$\frac{\sqrt{n}}{t_0} K (1 - \frac{\rho^2}{t_0^2(1-\lambda)^2})^{\frac{n-3}{2}} \approx \frac{\sqrt{n}}{t_0} \frac{\sqrt{n}\rho}{t_0} = \frac{\sqrt{n}}{t_0} \sqrt{\log K},$$

which finishes the proof of Theorem 2.8 and thus Theorem 1.5. $\square$

5. Improvements in some partial cases

In certain cases Theorem 1.1 may be improved and made prescize. Namely, we fix $p > 0$ and consider $\varphi(y) = \varphi_p(y) = \frac{|y|^p}{p}$ which corresponds to a measure γ_p with density $e^{-\frac{|y|^p}{p}}$. Such measures are log-concave for $p \geq 1$. They were considered in [20]. It was shown there, that for every convex body Q in $\mathbb{R}^n$,

$$\gamma_p(\partial Q) \lesssim n^{\frac{3}{4}-\frac{1}{p}}.$$

The definition of t_0 implies that $t_0 \approx n^{\frac{1}{p}}$ for the measures γ_p. Thus the above estimate can be rewritten:

$$\gamma_p(\partial Q) \lesssim \frac{\sqrt{n}}{t_0} n^{\frac{1}{4}}.$$

In particular, it was shown in [20] that for every convex body Q,

$$\gamma_p(\partial Q) \lesssim \max_{y\in\partial Q} \frac{\alpha(y)|y|^{\frac{p}{2}}+1}{|y|^{\frac{p}{2}-1}} \approx \max_{y\in\partial Q} \frac{\sqrt{n}}{t_0}(r(y)+1), \tag{39}$$

where, as before, $\alpha(y) = \cos(y, n_y)$ and $r(y) = \frac{\sqrt{n}}{t_0}\alpha(y)|y|$. Using the scheme from the proof of Theorem 2.7 we observe that for any polytope P with K facets, and for any $R > 0$,

$$\gamma_p(\partial Q) \lesssim \frac{\sqrt{n}}{t_0}(R + Ke^{-cR^2}). \tag{40}$$

Minimizing (40) in R, we get the following

Theorem 5.1.

$$\gamma_p(\partial P) \lesssim \frac{\sqrt{n}}{t_0}\sqrt{\log K} \approx n^{\frac{1}{2}-\frac{1}{p}}\sqrt{\log K}.$$

The above estimate is optimal since it coincides with the lower bound from Theorems 1.5 and 2.8.

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