RESEARCH STATEMENT

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Introduction

My research is devoted to exploring the geometry of convex bodies in high dimensions, using methods and ideas from probability, harmonic analysis, differential geometry and combinatorics.

One of my current interests is estimating marginals of probability distributions. Consider a simple-sounding question: *which hyperplane section of the unit cube in $\mathbb{R}^n$ has maximal $(n-1)$-dimensional volume?* K. Ball [7] showed that for every $n \geq 3$, the maximal section of the unit cube, up to the natural symmetry of the cube, is orthogonal to $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, ..., 0)$ and is of $(n-1)$-dimensional volume $\sqrt{2}$. Moreover, K. Ball showed in [8] that the $(n-k)$-dimensional volume of any k-codimensional section of the unit cube is bounded from above by $\sqrt{2}^k$. Recently, Rudelson and Vershynin [97] showed that an $\infty-$norm of a k-codimensional marginal of a product density with bounded coordinates is bounded from above by c^k, where c is an absolute constant. They conjectured that, in fact, $c = \sqrt{2}$, as in the case of the cube.

• In [72], G. Paouris, P. Pivovarov and myself obtained a sharp constant for the $\infty-$norm of k-codimensional marginals of a random vector X with independent entries. Our bound depends on the bounds on the densities of X_i, $i = 1, ..., n$ in an optimal way, and generalizes the results by Ball for a random vector uniformly distributed on the unit cube. In particular, it confirms the bound $\sqrt{2}^k$ for the densities conjectured in [97] by Rudelson and Vershynin.

I am also interested in combinatorial aspects of convexity. Consider the following question: *Given a closed convex body K in $\mathbb{R}^n$, how many translates of the interior of K are necessary to cover K in their union?*. Observe that if K is a cube, then one needs 2^n translates of its interior to cover every vertex. Hadwiger's illumination conjecture (see [43]) states that the minimal number of translates of $int(K)$ covering K (called the illumination number of K) is bounded from above by 2^n for every convex body K in $\mathbb{R}^n$. The best currently known upper bound is of order $4^n\sqrt{n}\log n$, and is due to Rogers [94]. See Boltyanski, Martini, Soltan [19], Bezdek [12], Bezdek, Khan [14] for a detailed discussion of the problem and a survey of known results.

• In [74], K. Tikhomirov and myself studied a "dilated" version of the illumination number, previously considered by Meir and Moser [84], Januszewski [45] and Naszódi [90]. Via a probabilistic scheme, we obtained the best currently known bound for it, which, in particular, recovers Rogers's bound for the classical illumination number.

Another project of mine is related to the Log-Minkowski conjecture, proposed by Lutwak: *Does the cone volume measure of a symmetric convex body determine it uniquely?* In particular, *given two symmetric polytopes P_1 and P_2 in $\mathbb{R}^n$ with faces $\{F_1^j\}_{j=1}^N$ and $\{F_2^j\}_{j=1}^N$ correspondingly, orthogonal to a fixed collection of vectors $\{u_j\}_{j=1}^N$, does $Vol_n(conv(0, F_1^j)) = Vol_n(conv(0, F_2^j))$ for every $j = 1, .., N$ imply that $P_1 = P_2$?* The Log-Minkowski problem is closely related to Log-Brunn-Minkowski inequality, conjectured in [21], and to the L_p-Minkowski existence problem, partially solved by Lutwak, Yang, Zhang [83]. The Log-Brunn-Minkowski conjecture states that the volume of the geometric average of a pair of symmetric convex bodies is bounded from below by the geometric average of their volumes. It is known to be true on the plane (see [21]) and for unconditional bodies (see [100]). Crucial contributions to the subject have been made by Böröczky, Lutwak, Yang, Zhang [21], [22], [23], Stancu [106], [107], Saraglou [100], [101], Huang, Liu, Xu [44] and others.

• In [71], Marsiglietti, Nayar, Zvavitch and myself found a relation between the conjectured Log-Brunn-Minkowski inequality and a dimensional Brunn-Minkowski inequality for log-concave measures, conjectured by Gardner and Zvavitch [38], thereby obtaining this inequality on the plane and for unconditional convex bodies. The said dimensional Brunn-Minkowski inequality is, essentially, a strengthening of log-concavity in the presence of symmetry.

• In [29], Colesanti, Marsiglietti and myself have established infinitesimal versions of Log-Brunn-Minkowski and dimensional Brunn-Minkowski inequalities, thereby obtaining a family of Poincare type inequalities for unconditional convex bodies. We proved the validity of Log-Brunn-Minkowski inequality for symmetric convex bodies close to a unit ball. We showed as well that the dimensional Brunn-Minkowski inequality holds for *not necessarily symmetric* convex bodies near a ball, in the case of rotation-invariant log-concave measures.

Minkowski's Theorem asserts that any centered measure on the unit sphere, which is not concentrated on the great subsphere, is the surface area measure of some convex body; moreover, a convex body is determined uniquely by its surface area measure.

• In [70], I proved an extension of Minkowski's existence theorem for measures with positive degree of concavity and positive degree of homogeneity, which implies, in particular, a weaker form of log-Minkowski problem for smooth and strictly convex sets.

The Shephard problem is the following question: *if the areas of projections of a convex body K in $\mathbb{R}^n$ are smaller then the areas of projections of a convex body L in $\mathbb{R}^n$ in every direction, does it imply that the volume of K is smaller then the volume of L?* The problem was posed by Shephard [105] and solved independently by Petty [93] and Schneider [102]. The answer is affirmative for $n = 2$ and negative for $n \geq 3$. See books by Koldobsky [55] and Koldobsky, Yaskin [57] for an introduction of Fourier analytic methods to Convexity, in particular, for the study of the Shephard problem and related questions.

In [70], I applied the extension of Minkowski's existence theorem to questions related to measure comparison of convex bodies, extending the solution of Shephard's problem to the measures with positive degree of concavity and positive degree of homogeneity.

During my Ph.D. studies, I have been working on isoperimetric type questions in high dimensions. The classical Isoperimetric inequality states that of all the sets of fixed volume the ball has the smallest surface area. Nowadays, dozens of generalizations and companions of this fact have been developed. An especially interesting one is the Gaussian Isoperimetric Inequality, obtained independently by Sudakov and Tsirelson [108] in 1974 and Borell [20] in 1975. It states that *among all the sets in $\mathbb{R}^n$ with prescribed Gaussian measure, half spaces have the smallest Gaussian surface area.* By Gaussian surface area here we understand the Gaussian density averaged against the surface measure of the set.

Mushtari and Kwapien asked the reverse question: *how large can the Gaussian perimeter of a convex set in $\mathbb{R}^n$ be*? In the literature this question has been referred to as the "reverse isoperimetric problem". Ball [6] proved in 1993 that the Gaussian perimeter of a convex set in $\mathbb{R}^n$ is bounded from above by $Cn^{\frac{1}{4}}$, where C is an absolute constant independent of the dimension. Nazarov [89] showed in 2003 that this bound is asymptotically exact. Further estimates for the special case of polynomial level set surfaces were provided by Kane [47].

In [65], [66] and [67] I studied the question of Mushtari and Kwapien for more general classes of measures.

• In [65], I considered the probability measure γ_p with density $C_{n,p}e^{-\frac{|x|^p}{p}}$, where $p > 0$. I have shown that the maximal perimeter of a convex set in $\mathbb{R}^n$ with respect to γ_p is of order $C(p)n^{\frac{3}{4}-\frac{1}{p}}$, where $C(p)$ is a constant independent of the dimension.

• In [66], I solved the reverse isoperimetric problem for the entire class of log-concave rotation invariant probability measures (roughly speaking, the measure is log-concave if the logarithm of its density is a concave function). The latter class includes all the measures γ_p when $p \geq 1$, and, in particular, the standard Gaussian measure (the case $p = 2$). It also includes the Lebesgue measure restricted to a ball.

• In [67], I carried out a more refined study of the maximal perimeter of polytopes in $\mathbb{R}^n$ depending on the number of their facets.

I am currently working on further questions related to all of my past work; they are discussed in detail in the end of each of the sections of the present research statement. In addition, most of the sections contain subsections dedicated to prospective graduate projects which I keep in mind in the case a graduate student expresses interest in undertaking research with me.

1. Bounding marginal densities and small ball inequalities.

In [7], Ball considered the following question: of all the $(n-1)$-dimensional slices of the unit cube Q_n, which slice has the largest $(n-1)$-dimensional volume? He proved that for all $u \in \mathbb{S}^{n-1}$ (the unit sphere in $\mathbb{R}^n$),

$$|Q_n \cap u^{\perp}|_{n-1} \leq \sqrt{2}. \tag{1}$$

This bound is sharp in every dimension, as it is attained for $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, ..., 0)$. Moreover, in [8], Ball proved that for each $(n-k)$-dimensional subspace H of $\mathbb{R}^n$,

$$|Q_n \cap H|_{n-1} \leq \min\left(\sqrt{2}^k, \left(\frac{n}{n-k}\right)^{\frac{n-k}{2}}\right). \tag{2}$$

The bound of $\sqrt{2}^k$ is sharp for all k such that $k \leq \frac{n}{2}$, and the bound $\left(\frac{n}{n-k}\right)^{\frac{n-k}{2}}$ is attained whenever $(n-k) \mid n$.

Ball's approach to cube slicing has been adapted to a variety of related problems, see Barthe [10], Gluskin [40], Koldobsky and König [59], Brzezinski [26]. Below we present a natural extension of the results from [7], [8] to a more general setting.

For a probability density f on $\mathbb{R}^n$ and a linear subspace E, the marginal density of f on E is given by

$$\pi_E(f)(x) = \int_{E^{\perp}+x} f(y)dy \quad (x \in E).$$

In other words, if $X \in \mathbb{R}^n$ is distributed according to f, then the density of the orthogonal projection $P_E X$ of X onto E is $\pi_E(f)$. Therefore, it is of interest to have the inequality $||\pi_E(f)||_{\infty} \leq C^k$, for an absolute constant C: in this case, the straightforward integral bound shows that for each $z \in E$, and for every $\epsilon > 0$,

$$P\left(|P_E X - z| \leq \epsilon\sqrt{k}\right) \leq (C\sqrt{2e\pi}\epsilon)^k. \tag{3}$$

The small ball inequality (3) is useful for various estimates in probability and analysis, particularly random matrices. For instance, Rudelson and Vershynin [97] considered the following class of probability densities on $\mathbb{R}^n$:

$$\mathcal{F}_n = \left\{ f(x) = \prod_{i=1}^{n} f_i(x_i) \,:\, ||f_i||_{\infty} \leq 1 = ||f_i||_1 \right\}. \tag{4}$$

They proved that if $f(x) \in \mathcal{F}$, then for any $k \in \{1, \ldots, n-1\}$, and any subspace E of dimension k, $||\pi_E(f)||_{\infty}^{1/k} \leq C$, where C is an absolute constant. Rudelson and Vershynin conjectured that the constant C should in fact be taken to be equal $\sqrt{2}$, to match Ball's result (2). In [72], Paouris, Pivovarov and myself confirmed this estimate, and a much stronger statement, applicable to a wider class of functions, was obtained.

Theorem 1.1 (Livshyts, Paouris, Pivovarov). *Let $1 \leq k < n$ and $f(x)$ be a bounded product probability density on $\mathbb{R}^n$ given via $f(x) = \prod_{i=1}^n f_i(x_i)$. Then for each $E \in G_{n,k}$ (a k-dimensional* *subspace of* $\mathbb{R}^n$*), there exists a collection of numbers* γ_i*,* $i = 1, ..., n$ *such that* $\gamma_i \in [0,1]$ *and* $\sum_{i=1}^n \gamma_i = k$*, and so that*

$$||\pi_E(f)||_\infty \leq \min\left(\left(\frac{n}{n-k}\right)^{\frac{n-k}{2}}, 2^{k/2}\right) \cdot \prod_{i=1}^n ||f_i||^{\gamma_i}_{L^\infty(\mathbb{R})}. \tag{5}$$

The techniques used in the proof of Theorem 1.1 include rearrangement inequalities, Fourier analysis, and Ball's Borell-Brascamp-Lieb inequality in geometric form (see [8]).

Another relevant study was conducted by Rogozin [96]: he showed that if θ is a unit vector with linear span $[\theta]$, then $||\pi_{[\theta]}(f)||_\infty \leq ||\pi_{[\theta]}(1_{Q_n})||_\infty$ for any f in the class $\mathcal{F}_n$. Therefore, for $k = 1$ one has a strong, pointwise version of Theorem 1.1 in the case when $f \in \mathcal{F}_n$. It turns out that the pointwise bound for the k-dimensional marginals of $f \in \mathcal{F}$ holds for all $k \in [1, n-1]$.

Proposition 1.2 (Livshyts, Paouris, Pivovarov)**.** *For every* $f \in \mathcal{F}_n$ *and for every* $E \in G_{n,n-k}$*,*

$$||\pi_E(f)||_\infty \leq |Q_n \cap E^\perp|.$$

Proposition 1.2 follows from a theorem of Kanter [50]. Moreover, a dual estimate is possible in the class of product densities on $\mathbb{R}^n$ supported on the unit cube Q_n via similar means. These statements and related results shall be presented in [73].

1.1. Future work and current projects.

My further joint work with Paouris and Pivovarov on the subject is focused on a number of related questions, in particular, the following problems.

Problem 1.3. *Given that a probability density* f *is unconditional (symmetric with respect to all coordinate hyperlpanes), find a good uniform bound for the infinity norms of its marginals.*

Some progress is possible for the class of unconditional densities which coincide with distribution functions on $\{x \in \mathbb{R}^n : x_i < 0\}$, with the help of ideas from [72] along with the techniques of mixing measures (popularized by Klartag in his breakthrough work [53]).

Another line of attack is to obtain rearrangement inequalities for functions $f(x) = F(f_1(x_1), ..., f_n(x_n))$, where $f_i(t)$ are bounded probability densities on $\mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}^+$ belongs to a class of functions including $F(a_1, ..., a_n) = \prod_{i=1}^n a_i$. The class of functions F should perhaps also include $F(a_1, ..., a_n) = a_1 + ... + a_n$ and $F(a_1, ..., a_n) = \max(a_1, ..., a_n)$, as certain steps in the proof can be verified in the latter cases.

2. Randomized covering and illumination.

Given a convex body K (a compact convex set with non-empty interior) in $\mathbb{R}^n$, and a point $x \in \partial K$, we say that a *direction* $u \in \mathbb{R}^n \setminus \{0\}$ *illuminates* x if there exists an $\epsilon > 0$ such that $x + \epsilon u \in int(K)$. We say that a point $p \in \mathbb{R}^n \setminus K$ illuminates $x \in \partial K$ if the direction $x - p$ illuminates x. A collection of points $p_1, ..., p_N \in \mathbb{R}^n \setminus K$ is said to illuminate K if every point $x \in \partial K$ is illuminated by at least one member of this collection. The *illumination number* $\mathcal{I}(K)$ is the cardinality of the smallest collection of points in $\mathbb{R}^n \setminus K$ illuminating K.

A well known illumination conjecture of H. Hadwiger [43], independently formulated by I. Gohberg and A. Markus, asserts that $\mathcal{I}(K) \leq 2^n$ for any n-dimensional convex body, with the equality attained for parallelotopes. The problem is known to be equivalent to the question whether every convex body can be covered by at most 2^n smaller homothetic copies of itself (see, for example, V. Boltyanski, H. Martini, P. S. Soltan, [19, Theorem 34.3]), K. Bezdek [12, Chapter 3], [13], K. Bezdek and M. A. Khan [14], Schramm [104].)

The best currently known upper bound for the illumination number, which follows from a classical covering argument of C.A. Rogers [94], is

$$\mathcal{I}(K) \leq (n \log n + n \log\log n + 5n) \frac{\mathrm{Vol}_n(K-K)}{\mathrm{Vol}_n(K)} \tag{6}$$

(see, for example, K. Bezdek [12, Theorem 3.4.1]). Using the estimate of $\mathrm{Vol}_n(K-K)$ due to C.A. Rogers and G.C. Shephard [95], we get $\mathcal{I}(K) \le (1+o(1))\binom{2n}{n} n \log n$. Moreover, for a centrally-symmetric K we clearly have $\mathrm{Vol}_n(K-K) = 2^n \mathrm{Vol}_n(K)$, whence $\mathcal{I}(K) \le (1+o(1))2^n n \log n$.

In [74], Tikhomirov and the myself proved the following.

Proposition 2.1 (Livshyts, Tikhomirov). *Let n be a sufficiently large positive integer, K be a convex body in $\mathbb{R}^n$ with the origin in its interior and let numbers $(\lambda_i)_{i=1}^m$ satisfy $\lambda_i \in (e^{-n}, 1)$ $(i = 1, 2, \dots, m)$ and*

$$\sum_{i=1}^m {\lambda_i}^n \ge (n \log n + n \log\log n + 4n) \frac{\mathrm{Vol}_n(K-K)}{\mathrm{Vol}_n(K)}.$$

For each i, let X_i be a random vector uniformly distributed inside the set $K - \lambda_i K$, so that $X_1, X_2, \dots, X_m$ are jointly independent. Then the random collection of translates $\{X_i + \lambda_i K\}_{i=1}^m$ covers K with probability at least $1 - e^{-0.3n}$.

For a convex body K in $\mathbb{R}^n$, define $f_n(K)$ to be the least positive number such that for any sequence (λ_i) $(\lambda_i \in [0, 1))$ with

$$\sum {\lambda_i}^n > f_n(K),$$

there are points $x_i \in \mathbb{R}^n$ such that the collection of homothets $\{\lambda_i K + x_i\}$ covers K. Estimates on $f_n(K)$ were previously obtained by Meir, Moser [84], Januszewski [45] and Naszódi [90].

In [74], Tikhomirov and myself obtained the best currently known upper bound for this quantity, using proposition 2.1.

Theorem 2.2 (Livshyts, Tikhomirov). *Let n be a (large enough) positive integer and K be a convex body in $\mathbb{R}^n$. Then, with the quantity $f_n(K)$ defined above, we have*

$$f_n(K) \le (n \log n + n \log\log n + 5n) \frac{\mathrm{Vol}_n(K-K)}{\mathrm{Vol}_n(K)}.$$

In particular, Theorem 2.2 recovers the best known bound for the classical illumination number.

2.1. **Further work and current problems.** In addition to the actual illumination conjecture, my current joint projects with Tikhomirov include the following.

Problem 2.3. *1. Show the sharp upper bound for the illumination number of random polytopes. 2. Show that the illumination number of a convex body which is close to a parallelotope in Banach-Mazur distance, but is not a parallelotope itself, is bounded from above by $2^n - 1$.*

A possible line of attack for this problem is based on the following observations. Firstly, the cardinality of the minimal collection of points illuminating K equals to the cardinality of the minimal collection of directions illuminating K. A point $x \in \partial K$ can be potentially illuminated by the set of directions $s(x) = \{u \in \mathbb{S}^{n-1} : \langle u, v\rangle < 0\ \forall v \in \nu_K(x)\}$, where the Gauss map $\nu_K : \partial K \to \mathbb{S}^{n-1}$ corresponds $x \in \partial K$ to the set of normal vectors at x. Given a polytope P, consider a set of spherical caps $S_1, ..., S_N$ illuminating its vertices $x_1, ..., x_N$ correspondingly; the problem of illuminating P with M light sources is equivalent to the problem of selecting M representatives on the sphere from the collection $S_1, ..., S_N$. Selecting those representatives randomly according to the density $\sum 1_{S_i}(u) : \mathbb{S}^{n-1} \to [0, \infty)$ may be a fruitful idea. In particular, some estimates are possible under the assumption that the total area of S_i is large. By De Gua's formula, the sum of solid angles of a polytope is proportional to $C(n, N)$ minus the sum of the perimeters of its dual cones (see, e.g. Pak [91]), hence it is of interest to bound from above the total perimeter of the partition of the unit sphere by normal cones of a polytope.

Problem 2.4. *Let P be a polytope in $\mathbb{R}^n$. Let the unit sphere $\mathbb{S}^{n-1}$ be partitioned into polygonal spherically convex caps $\nu_1(P), ..., \nu_N(P)$, where $\nu_i(P)$ is the set of normal vectors at the vertex x_i*

of P. Show that there exists a linear transformation $T:\mathbb{R}^n\to\mathbb{R}^n$ such that $\sum_{i=1}^N p(\nu_i(TP))$ (the total sum of the perimeters of $\nu_i(TP)$) is bounded from above by some constant $f(N,n)$, and this bound is optimal when P is a simplex.

This problem could be viewed as a spherical analogue of K. Ball's reverse isoperimetric inequality [9], and I intend to incorporate some of the methods from [9] in my consideration of this problem.

2.2. **Prospective projects for graduate students.** In [4], Artstein-Avidan and Raz obtained estimates on weighted covering and separation numbers. They considered a model, where instead of the classical covering, one covers a convex body with weighted copies of itself. The following problem would be of interest.

Problem 2.5. *Show tight bounds on a weighted-dilated covering number, and on a weighted-dilated separation number.*

Additionally, the following problem arose in my discussion with Ben Ide, a graduate student at Georgia Tech.

Problem 2.6. *Fix $R>0$ and a positive integer $n\geq 1$. Let K be a convex body. Find an upper bound on the number of sources of light which would be necessary to illuminate the boundary of K, given that the distance to the sources cannot exceed R.*

A similar problem was considered and solved by Litvak and Bezdek [15]: instead of bounding the euclidean distance to the sources they considered bounding the average distance to the sources in the metric of K.

3. On the Log-Brunn-Minkowski inequality and Log-Minkowski uniqueness problem.

The classical Brunn-Minkowski inequality states that for every pair of Borel sets A and B in $\mathbb{R}^n$, and for every $\lambda\in[0,1]$, one has

$$|\lambda A+(1-\lambda)B|^{\frac{1}{n}}\geq\lambda|A|^{\frac{1}{n}}+(1-\lambda)|B|^{\frac{1}{n}}, \tag{7}$$

or, equivalently

$$|\lambda A+(1-\lambda)B|\geq|A|^{\lambda}|B|^{1-\lambda}. \tag{8}$$

See Gardner [35] or Schneider [103] for more details; see also Figalli, Maggi, Pratelli [33], Figalli, Jerison [34] for a sharpening of (7), effective in the case when A and B have very different shape. Very recently I have been working with Michael Damron on questions related to the geometry of the limiting shape of the Eden model from first passage percolation, and the Brunn-Minkowski inequality comes very useful.

Set σ_K to be the surface area measure of K, that is, the push forward of the $(n-1)$-dimensional Hausdorff measure on ∂K to $\mathbb{S}^{n-1}$ under the Gauss map. If the surface area measure has a density, then this density is called the curvature function and is denoted $f_K(u)$.

The support function $h_K:\mathbb{R}^n\to\mathbb{R}^+$ of a convex body K is defined as

$$h_K(x)=\max_{y\in K}\langle x,y\rangle.$$

For a unit vector x, the support function $h_K(x)$ is the distance from the origin to the support hyperplane of K orthogonal to x.

A measure γ on $\mathbb{R}^n$ is called log-concave if for any Borel measurable sets $A,B\subset\mathbb{R}^n$ and for any $\lambda\in[0,1]$,

$$\gamma(\lambda A+(1-\lambda)B)\geq\gamma(A)^{\lambda}\gamma(B)^{1-\lambda}.$$

Given a scalar $\lambda \in [0,1]$, and a pair of convex bodies K and L containing the origin in their interior, with support functions h_K and h_L, respectively, their geometric average is defined as follows:

$$K^\lambda L^{1-\lambda} := \{x \in \mathbb{R}^n \,:\, \langle x, u\rangle \leq h_K^\lambda(u) h_L^{1-\lambda}(u) \;\forall u \in \mathbb{S}^{n-1}\}. \tag{9}$$

Recall that a set is called symmetric if $x \in K$ implies $-x \in K$. The following conjecture is widely known as log-Brunn-Minkowski conjecture (see Böröczky, Lutwak, Yang, Zhang [21], [22], [23]).

Conjecture 3.1 (Böröczky, Lutwak, Yang, Zhang)**.** *Let $n \geq 2$ be an integer. Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Then*

$$|K^\lambda L^{1-\lambda}| \geq |K|^\lambda |L|^{1-\lambda}. \tag{10}$$

Note that (10) is stronger then (8) since $K^\lambda L^{1-\lambda} \subset \lambda K + (1-\lambda)L$. Böröczky, Lutwak, Yang and Zhang [21] showed that Conjecture 3.1 holds for $n = 2$. Saroglou [100] and Cordero, Fradelizi, Maurey [31] proved that (10) is true when the sets K and L are unconditional (that is, they are symmetric with respect to every coordinate hyperplane). Saroglou showed as well [101] that the validity of Conjecture 3.1 would imply the same statement for every log-concave even measure γ on $\mathbb{R}^n$: for every convex symmetric $K, L \subset \mathbb{R}^n$ and for every $\lambda \in [0,1]$,

$$\gamma(K^\lambda L^{1-\lambda}) \geq \gamma(K)^\lambda \gamma(L)^{1-\lambda}. \tag{11}$$

The Log-Brunn-Minkowski conjecture is stronger then the B-conjecture, another well known problem in convexity, formulated by Banaszczyk and popularized by Latała [60].

Conjecture 3.2 (B-conjecture)**.** *Let $n \geq 2$ be an integer, and let γ be a log-concave even measure on $\mathbb{R}^n$. Then for every symmetric convex body $K \subset \mathbb{R}^n$, the function $t \to \gamma(e^t K)$ is log-concave on $\mathbb{R}^+$.*

The B-conjecture was proved in the case of Gaussian measure by Cordero-Erausquin, Fradelizi and Maurey [31]. Their results were extended by Livne Bar-on [64].

Minkowski's existence theorem guarantees that every barycentered measure on $\mathbb{S}^{n-1}$ which is not supported on any great subsphere is a surface area measure for some convex body; moreover, a convex body is determined uniquely by its surface area measure.

The cone volume measure of a C^2-smooth strictly-convex body K in $\mathbb{R}^n$ is the measure on the unit sphere with density $\frac{1}{n} h_K(u) f_K(u)$, $u \in \mathbb{S}^{n-1}$. Integrating this measure over the whole sphere, one recovers the volume of K. One of the most important applications for a positive resolution to Conjecture 3.1 would be the positive resolution to the following conjecture (see Böröczky, Lutwak, Yang and Zhang [21] for the reasoning).

Conjecture 3.3. *Let K and L be symmetric strictly-convex C^2-smooth bodies in $\mathbb{R}^n$ such that $h_K(u) f_K(u) = h_L(u) f_L(u)$ for every $u \in \mathbb{S}^{n-1}$. Then $K = L$.*

This conjecture is equivalent to the uniqueness of a solution of a Monge-Ampere type equation in the class of even support functions, since the curvature function can be expressed in Aleksandrov's form in terms of the support function. We remark, that without the smoothness and strict convexity assumptions, the statement of the Conjecture 3.3 does not hold: a counter example is parallelograms with parallel sides on the plane (see Stancu [106]).

For $p \in \mathbb{R}$, the L_p surface area measure of a convex body is the measure on the sphere given by $d\sigma_{p,K}(u) = h_K^{1-p}(u) d\sigma_K(u)$. An extension of Minkowski's Theorem to the cases when $p \neq 1$, called *L_p-Minkowski problem* is open in general, and includes Conjecture 3.3. Numerous papers have been published on the subject, some of them are: Böröczky, Lutwak, Yang, Zhang [21], [22], [23], Huang, Liu, Xu [44], Chou, Wang [27], Naor [88].

3.1. **Minkowski's Theorem for measures.** First, we introduce a natural analogue of the surface area measure in the case when the underlying measure is not Lebesgue.

Definition 3.4. *Let K be a convex body and ν_K be its Gauss map. Let μ be a measure on $\mathbb{R}^n$ with density $g(x)$ continuous on its support. Define $\sigma_{\mu,K}$ on $\mathbb{S}^{n-1}$, a surface area measure of K with respect to μ, as follows: for every Borel set $\Omega \subset \mathbb{S}^{n-1}$, let*

$$\sigma_{\mu,K}(\Omega) = \int_{\nu_K^{-1}(\Omega)} g(x) dH_{n-1}(x), \tag{12}$$

where H_{n-1} stands for the $(n-1)-$dimensional Hausdorff measure on ∂K, and $\nu_K^{-1}(\Omega)$ stands for the full pre-image of Ω under ν_K.

When μ is the standard Lebesgue measure, the measure $\sigma_{\mu,K}$ coincides with σ_K, the classical surface area measure.

Let $p \in (0, +\infty)$. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is $p-$concave if $f^p(x)$ is a concave function on its support. Let $r \in (-\infty, +\infty)$. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is $r-$ homogenous if for every $a > 0$ and for every $x \in \mathbb{R}^n$ we have $f(ax) = a^r f(x)$.

E. Milman and L. Rotem [86] considered the class of measures on $\mathbb{R}^n$ with densities that have a positive degree of homogeneity and a positive degree of concavity and studied their isoperimetric properties. Note that such measures are necessarily supported on convex cones.

In [70] I proved an extension of Minkowski's existence theorem to the class of surface area measures with respect to measures with positive degree of concavity and positive degree of homogeneity.

Theorem 3.5 (Livshyts)**.** *Let μ on $\mathbb{R}^n$ be a measure with an even density $g(x)$ continuous on its support, with positive degree of homogeneity, and such that its restriction to some half space has positive degree of concavity. Let φ be an arbitrary even measure on $\mathbb{S}^{n-1}$, not supported on any great subsphere, such that supp$(\varphi) \subset$ int(supp$(g)) \cap \mathbb{S}^{n-1}$. Then there exists a symmetric convex body K in $\mathbb{R}^n$ such that*

$$d\sigma_{K,\mu}(u) = d\varphi(u).$$

Moreover, such convex body is determined uniquely up to a set of $\mu-$measure zero.

Theorem 3.5 provides a new view on the Brunn-Minkowski theory, and some of its consequences illuminate the understanding of the classical surface area measure itself. In particular, it implies the following proposition, which can be viewed as a weaker version of the assertion of Conjecture 3.3.

Proposition 3.6 (Livshyts)**.** *Let K and L be two symmetric smooth strictly convex bodies in $\mathbb{R}^n$ with support functions h_K and h_L and curvature functions f_K and f_L such that*

$$\frac{\partial h_K(x)}{\partial x_1} f_K(x) = \frac{\partial h_L(x)}{\partial x_1} f_L(x)$$

for every $x \in \mathbb{S}^{n-1}$. Then $K = L$.

3.2. **Infinitesimal versions.** Gardner and Zvavitch [38] conjectured another strengthening of Brunn-Minkowski inequality in the case of the Gaussian measure; we shall state it in a more general form which is believed to be natural.

Conjecture 3.7 (Gardner, Zvavitch (generalized))**.** *Let $n \geq 2$ be an integer. Let γ be a log-concave symmetric measure (that is, $\gamma(A) = \gamma(-A)$ for every measurable set A) on $\mathbb{R}^n$. Let K and L be symmetric convex bodies in $\mathbb{R}^n$. Then*

$$\gamma(\lambda K + (1-\lambda)L)^{\frac{1}{n}} \geq \lambda \gamma(K)^{\frac{1}{n}} + (1-\lambda)\gamma(L)^{\frac{1}{n}}. \tag{13}$$

In [71], jointly with Marsiglietti, Nayar and Zvavitch, I showed that the Log-Brunn Minkowski inequality (10) implies the conjecture of Gardner and Zvavitch for any even log-concave measure and for every pair of symmetric convex sets, and derived the following.

Theorem 3.8. *(Livshyts, Marsiglietti, Nayar, Zvavitch)*

(1) *Let γ on $\mathbb{R}^n$, $n \geq 2$, be an unconditional log-concave measure on $\mathbb{R}^n$ and let K and L be unconditional convex sets in $\mathbb{R}^n$. Then for every $\lambda \in [0,1]$,*

$$\gamma(\lambda K + (1-\lambda)L)^{\frac{1}{n}} \geq \lambda\gamma(K)^{\frac{1}{n}} + (1-\lambda)\gamma(L)^{\frac{1}{n}}.$$

(2) *If γ is a symmetric log-concave measure on $\mathbb{R}^2$ and K and L are symmetric convex sets in $\mathbb{R}^2$, then for every $\lambda \in [0,1]$,*

$$\gamma(\lambda K + (1-\lambda)L)^{\frac{1}{2}} \geq \lambda\gamma(K)^{\frac{1}{2}} + (1-\lambda)\gamma(L)^{\frac{1}{2}}.$$

In [29], jointly with Colesanti and Marsiglietti, I have developed an infenitesimal version of the Log-Brunn-Minkowski inequality (10). Additionally it was shown in [29] that the Log-Brunn Minkowski inequality holds for log-concave rotation invariant measures and symmetric convex sets which are close to a ball.

Theorem 3.9 (Colesanti, Livshyts, Marsiglietti)**.** *Let γ be a rotation invariant log-concave measure on $\mathbb{R}^n$. Let $R \in (0,\infty)$. Let $\varphi \in C^2(\mathbb{S}^{n-1})$ be even and strictly positive. Then there exists a sufficiently small $a > 0$ such that for every $\epsilon_1, \epsilon_2 \in (0,a)$ and for every $\lambda \in [0,1]$, one has*

$$\gamma(K_1^{\lambda}K_2^{1-\lambda}) \geq \gamma(K_1)^{\lambda}\gamma(K_2)^{1-\lambda},$$

where K_1 is the convex set with the support function $h_1 = R\varphi^{\epsilon_1}$ and K_2 is the convex set with the support function $h_2 = R\varphi^{\epsilon_2}$.

It was also shown in [29] that the conjecture of Gardner and Zvavitch holds for log-concave rotation invariant measures and convex sets which are close to a ball (not necessarily symmetric).

3.3. **Future work and current projects.** There are three main directions of my investigations related to the present section.

1. Cone volume measure uniqueness. In view of Proposition 3.6, the following problem would be of great interest, since the positive answer would imply the positive answer to Conjecture 3.3.

Problem 3.10. *Let K and L be C^2-smooth symmetric strictly-convex bodies. Suppose $h_K(u)f_K(u) = h_L(u)f_L(u)$ for all $u \in \mathbb{S}^{n-1}$. Show that*

$$\frac{\partial h_K(u)}{\partial u_1}f_K(u) = \frac{\partial h_L(u)}{\partial u_1}f_L(u).$$

A possible line of attack for this problem would be to consider the following vector field on the unit sphere:

$$a(u) = \nabla h_K(u)f_K(u) - \nabla h_L(u)f_L(u).$$

If $h_K(u)f_K(u) = h_L(u)f_L(u)$ for all $u \in \mathbb{S}^{n-1}$, then this vector field is tangent in every point (due to 1-homogeneity of a support function). Additionally, it is smooth, due to the smoothness assumption on K and L. By the Hairy Ball Theorem and symmetry, there exist two points u_0 and $-u_0$ on the sphere such that $a(u_0) = a(-u_0) = 0$. To solve Conjecture 3.3 it would suffice to show that $a(u) = 0$ for all $u \in \mathbb{S}^{n-1}$; moreover, my results from [70] imply that it would be enough to show that there exists a unit vector $v \in \mathbb{S}^{n-1}$ such that $a(u) \perp v$ for all $u \in \mathbb{S}^{n-1}$.

In addition, Colesanti and myself have recently proved the local positive resolution of Conjecture 3.3 near a ball; the argument heavily uses Theorem 3.9, and shall be presented in [30].

Another question of interest is:

Problem 3.11. *Find an analogue of Minkowski's existence theorem for the standard Gaussian measure.*

Several results obtained in [70] may prove useful for this problem.

Various notions and results from classical convex geometry and Brunn-Minkowski theory have been recently generalized to the setting of lattice polytopes (see Alexander, Henk, Zvavitch [3], Gardner, Gronchi, Zong, [37], Gardner, Gronchi [36], Jochemko, Sanyal, [46], Ryabogin, Yaskin, Zhang [99], Zhang [110].)

Problem 3.12. *Find an analogue of Minkowski's existence theorem for lattice polytopes.*

The collection of tools which may be useful for this problem includes the tight discrete analogue of Brunn-Minkowski inequality proved by Gardner, Gronchi [36] and the notion and study of discrete mixed volumes by Johemko, Sanyal [46].

2. The Log-Brunn-Minkowski and Brunn-Minkowski inequalities via harmonic analysis. In [29], an infinitesimal version of the Log-Brunn-Minkowski inequality was obtained.

A corresponding infinitesimal Brunn-Minkowski inequality for Lebesgue measure was obtained by Colesanti in [28]. Both of the inequalities are Poincare-type inequalities on the unit sphere. For more on the Poincare type inequalities related to the Brunn-Minkowski inequality see Kolesnikov, E. Milman [58].

Problem 3.13. *Prove the infinitesimal versions of the Log-Brunn-Minkowski and Brunn-Minkowski inequalities directly, using spherical harmonics and other harmonic-analytic tools on the sphere.*

In the case $n = 2$, the infinitesimal Brunn-Minkowski inequality (and hence, the Brunn-Minkowski inequality on the plane for convex bodies) can be proved using nothing but Fourier expansions of periodic functions and their properties (see [29], section 2.)

3. A randomized Log-Brunn-Minkowski inequality. Another direction of my research is obtaining a randomized version of the Log-Brunn-Minkowski inequality. The following proposition follows from the results obtained by Saraglou [101].

Proposition 3.14. *Let $n \times n$ matrix A be a positive definite diagonal matrix and let an $n \times n$ matrix B be positive definite. Suppose the inequality*

$$\int_{\mathbb{R}^n} \langle Ex, x\rangle^2 d\mu(x) - \left(\int_{\mathbb{R}^n} \langle Ex, x\rangle d\mu(x)\right)^2 \leq \int_{\mathbb{R}^n} \langle (E^2 + E^t E)x, x\rangle d\mu(x), \tag{14}$$

holds for all n, A and B, where $E = B^{-1}AB$ and

$$d\mu(x) = \frac{1}{\int_{BQ} e^{-\frac{|x|^2}{2}} dx} 1_{BQ}(x) e^{-\frac{|x|^2}{2}} dx.$$

Then the Log-Brunn-Minkowski conjecture is true in every dimension.

Popescu, Ivanishvili, Saraglou and myself have shown that if E in (14) is taken to be a random matrix with i.i.d. entries, then the expectation of the right hand side of (14) is non-positive. To obtain it we have used a sharpening of Poincare inequality proved by Cordero-Erausquin, Fradelizi and Maurey [31]. Currently, we are working on the following

Problem 3.15. *Obtain concentration properties for the inequality (14), with an appropriate random model.*

3.4. **Prospective projects for graduate students.**

Problem 3.16. *Show the validity of the infinitesimal version of the Log-Brunn-Minkowski inequality on the plane* directly. *That is, prove that for every pair of* ***even***, *$[-\pi, \pi]-$periodic smooth functions ψ and h, such that $h + \ddot{h} > 0$ and $h > 0$, one has*

$$\left(\int h^2 - \dot{h}^2\right)\left(\int \psi^2 - \dot{\psi}^2 + \psi^2 \frac{h + \ddot{h}}{h}\right) \leq 2\left(\int h\psi - \dot{h}\dot{\psi}\right)^2. \tag{15}$$

This problem should be achievable via the same Fourier-analytic techniques, as was used for the proof of Brunn-Minkowski inequality on the plane in [29]. It could be useful to consider $\varphi = \psi h$ in place of ψ, as then the Fourier series of a product could be written in terms of convolution. I think that such direct proof, omitting the use of other inequalities, would illuminate the nature of the Log-Brunn-Minkowski conjecture.

4. Measure comparison and Shephard's problem for measures.

A projection of a convex body K to a subspace $u^\perp$ shall be denoted by $K|u^\perp$:

$$K|u^\perp = \{x \in u^\perp \, : \, \exists t \in \mathbb{R} \, s.t. \, x + tu \in K\}.$$

The Shephard problem (see Shephard [105]) is the following question: *given symmetric convex bodies K and L such that for every $u \in \mathbb{S}^{n-1}$*

$$|K|u^\perp|_{n-1} \leq |L|u^\perp|_{n-1}, \tag{16}$$

does it follow that $|K|_n \leq |L|_n$? The problem was solved independently by Petty [93] and Schneider [102], who showed that the answer is affirmative if $n \leq 2$ and negative if $n \geq 3$. Ball [5] proved that (16) implies that $|K| \leq \sqrt{n}|L|$, for every dimension n. See also Goodey and Zhang [41], Koldobsky, Ryabogin and Zvavitch [56], Ryabogin and Zvavitch [98].

In [70] I introduced a natural analogue of the Lebesgue measure of projection of a symmetric convex body to other measures. Recall the Cauchy projection formula, relating the area of a projection of K to the curvature function $f_K(u)$ (see, for example, Koldobsky [55]):

$$|K|\theta^\perp| = \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\langle u, \theta \rangle| f_K(u) du.$$

This formula helps to understand the next definition better.

Definition 4.1. *Let μ be a measure on $\mathbb{R}^n$ with density g continuous on its support, and let K be a convex body. Consider a unit vector $\theta \in \mathbb{S}^{n-1}$. Define the following function on the cylinder $\mathbb{S}^{n-1} \times [0, 1]$:*

$$p_{\mu,K}(\theta, t) := \frac{n}{2} \int_{\mathbb{S}^{n-1}} |\langle \theta, u \rangle| d\sigma_{\mu,tK}(u). \tag{17}$$

Here $\sigma_{\mu,tK}$ stands for the surface area measure of tK with respect to μ, as was defined in (12). We also consider $\mu -$ projection function on the unit sphere:

$$P_{\mu,K}(\theta) := \int_0^1 p_{\mu,K}(\theta, t) dt. \tag{18}$$

In the particular case of Lebesgue measure λ we have $P_{\lambda,K}(\theta) = |K|\theta^\perp|_{n-1}$. The good thing about $p_{\mu,K}(\theta, t)$ is that, as in the Lebesgue case, it is the Fourier transform of $\sigma_{\mu,tK}$, extended to $\mathbb{R}^n$ with the appropriate degree of homogeneity. Hence the inversion formula, a certain Parseval's identity, and other now classical Fourier-analytic techniques in Convexity, largely developed by Koldobsky, is applicable (see, e.g., [55], Koldobsky, Yaskin [57], Koldobsky, Ryabogin, Zvavitch [56], and the references therein).

A convex body is called a projection body if the Fourier transform of its (appropriately extended) support function is non-negative. The classical Shephard problem has positive solution if and only if the body L in (16) is a projection body (see Koldobsky [55], Chapter 8). In [70] I proved a more general result.

Theorem 4.2 (Livshyts)**.** *Fix $n \geq 1$, and consider $g : \mathbb{R}^n \to \mathbb{R}^+$, a function with a positive degree of concavity and a positive degree of homogeneity. Let μ be the measure on $\mathbb{R}^n$ with density $g(x)$.*

1. *Let K and L be symmetric strictly convex bodies, and let L additionally be a projection body. Assume that for every $\theta \in \mathbb{S}^{n-1}$ we have*

$$P_{\mu,K}(\theta) \leq P_{\mu,L}(\theta).$$

Then $\mu(K) \leq \mu(L)$.

(2) *If in addition we assume that the support of* g *is a half-space, then for each symmetric convex body* L *which is not a projection body, there exists a symmetric convex body* K *such that for every* $\theta \in \mathbb{S}^{n-1}$ *we have*

$$P_{\mu,K}(\theta) \leq P_{\mu,L}(\theta),$$

but $\mu(K) > \mu(L)$.

4.1. **Future work and current projects.** A number of results from [70] were obtained in a more general case then for measures with positive degree of concavity and homogeneity. Therefore, the following problem is of interest.

Problem 4.3. *Fix* $n \geq 1$*, and let measure* γ *be a standard Gaussian measure on* $\mathbb{R}^n$*. Let* K *and* L *be symmetric convex bodies, and let* L *additionally be a projection body. Assume that for every* $\theta \in \mathbb{S}^{n-1}$ *and for every* $t \in [0,1]$*, we have*

$$p_{\gamma,K}(\theta,t) \leq p_{\gamma,L}(\theta,t).$$

Show that $\gamma(K) \leq \gamma(L)$.

The statement would not hold if we assume just the inequality $P_{\gamma,K}(\theta) \leq P_{\gamma,L}(\theta)$, as it fails for symmetric strips. But I have no counterexamples for the refined version.

4.2. **Prospective projects for graduate students.** The approach to measure comparison via generalized projections, introduced in [70], generates questions about natural extensions of well-known related results to the case of, at least, measures with positive degree of homogeneity and positive degree of concavity.

Problem 4.4. *Extend the notion of a projection to a subspace of dimension* k*, where* $k = 1, ..., n$*, for measures other then the Lebesgue measure. Obtain an analogue of the multi-dimensional Shephard problem, solved by Goodey and Zhang* [41]*, for measures with a positive degree of concavity and a positive degree of homogeneity.*

Very recently, Giannopoulos and Koldobsky [39] answered a question of V. Milman: they showed, in particular, that for an arbitrary convex body K in $\mathbb{R}^n$ and a compact set D in $\mathbb{R}^n$, the inequality

$$|K|u^{\perp}| \leq |D \cap u^{\perp}|$$

for all $u \in \mathbb{S}^{n-1}$ implies that $|K| \leq |D|$.

Problem 4.5. *Extend the results of Giannopoulos and Koldobsky* [39] *to the case of measures with a positive degree of concavity and a positive degree of homogeneity.*

Another interesting question would be an extension of a result due to Ball [5].

Problem 4.6. *Given* μ *on* $\mathbb{R}^n$ *positively concave and homogenuous, find a good constant* $C(n)$*, such that* $P_{\mu,K}(\theta) \leq P_{\mu,L}(\theta)$ *for all* $\theta \in \mathbb{S}^{n-1}$ *implies* $\mu(K) \leq C(n)\mu(L)$ *for all convex bodies in* $\mathbb{R}^n$.

Let K and L be origin-symmetric convex integer polytopes in $\mathbb{R}^n$. A discrete analogue of the Aleksandrov projection problem is the following question: if for a vector with integer coordinates, the projections of K and L on $u^{\perp}$ have the same number of points, does it imply that $K = L$? A negative answer in the planar case was given by Ryabogin, Yaskin and Zhang [99], and a positive answer under some additional assumptions was provided by Zhang [110].

Problem 4.7. *Find a counterexample for the Aleksandrov projection problem in three dimensional case.*

I believe that this problem is possible to solve using computer programing. Recently, I suggested this problem to Ben Ide, a graduate student at GaTech, and he has started working on it.

5. Isoperimetric and reverse isoperimetric inequalities for log-concave measures.

The geometry of Log-concave measures has been studied intensively in recent years. For the background and numerous interesting properties, see, for example, results of Eldan [32], Guedon, Milman [42], Kannan, Lovasz, Simonovits [49], Klartag [51], [52], Milman [54], Pajor [87], Ledoux [62], Adamczak, Latała, Litvak, Tomczak-Jaegermann [1], [2], Paouris [92].

A measure γ is called rotation invariant if for every rotation T and for every set A, $\gamma(TA) = \gamma(A)$. Log-concave rotation invariant measures have been studied, in particular, by Bobkov in [16], [17], [18]. Examples of log-concave rotation invariant probability measures include the Standard Gaussian Measure γ_2 and Lebesgue measure restricted to a ball.

The surface area of a convex set Q with respect to a measure γ in the metric of a convex body L is defined to be

$$\gamma_1(Q, L) = \liminf_{\epsilon\to+0} \frac{\gamma((Q+\epsilon L)\backslash Q)}{\epsilon}.$$

Mostly, we shall talk about the usual surface area of Q with respect to γ, denoted by $\gamma^+(\partial Q) = \gamma_1(Q, B_2^n)$, where B_2^n is the unit euclidean ball in $\mathbb{R}^n$.

5.1. Dual isoperimetric inequality for a wide class of measures.

In [70] I proved the following.

Proposition 5.1 (Livshyts). *Let measure γ be log-concave. Then for every pair of Borel sets K and L such that $\gamma(K) = \gamma(L)$, one has*

$$\gamma_1(K, L) \geq \gamma_1(K, K).$$

I am currently working on the equality cases and applications of Proposition 5.1, which shall be presented in [69].

5.2. Reverse isoperimetric inequalities for log-concave rotation invariant measures.

Following a question raised by Mushtari and Kwapien, Ball [6] and Nazarov [89] obtained sharp bounds on the maximal perimeter of a convex set in $\mathbb{R}^n$ with respect to a Gaussian measure. Those bounds were later used by Bentkus [11] for the Central limit theorem in $\mathbb{R}^n$.

I have studied questions related to the surface area of a convex set in $\mathbb{R}^n$ with respect to log-concave rotation invariant probability measures. In [66] I proved the following theorem.

Theorem 5.2 (Livshyts). *Fix $n \geq 2$. Let γ be a log-concave rotation invariant probability measure on $\mathbb{R}^n$. Consider a random vector X in $\mathbb{R}^n$ distributed with respect to γ. Then*

$$\max_Q \gamma^+(\partial Q) \approx \frac{\sqrt{n}}{\sqrt{\mathbb{E}|X|}\sqrt[4]{Var|X|}},$$

where $\mathbb{E}|X|$ and $Var|X|$ stand for the expectation and the variance of $|X|$ respectively. The maximum runs over the class of convex sets in $\mathbb{R}^n$. The notation "$\approx$" means that the equality holds up to an absolute multiplicative constant.

Earlier, in [65] I considered the measures γ_p with densities $C_{n,p}e^{-\frac{|x|^p}{p}}$, where $p > 0$ and $C_{n,p}$ is the normalizing constant, and proved that $c(p)n^{\frac{3}{4}-\frac{1}{p}} \leq \max_Q \gamma_p(\partial Q) \leq C(p)n^{\frac{3}{4}-\frac{1}{p}}$, where $c(p)$ and $C(p)$ are constants depending on p only, and the maximum runs over the class of convex sets in $\mathbb{R}^n$. It is a direct consequence of Theorem 5.2 in the case of $p \geq 1$, but not $p < 1$.

Additionally, I worked on the reverse isoperimetric questions for the class of convex polytopes in $\mathbb{R}^n$ with K facets. In [67] I proved the following theorem.

Theorem 5.3 (Livshyts). *Fix $n \geq 2$. Let γ be a log-concave rotation invariant probability measure on $\mathbb{R}^n$. Consider a random vector X in $\mathbb{R}^n$ distributed with respect to γ. Fix a positive integer*

$K \in [1, e^{c\frac{\mathbb{E}|X|}{\sqrt{Var|X|}}}]$. *Let P be a convex polytope in $\mathbb{R}^n$ with K facets. Then*

$$\gamma^+(\partial P) \leq C\frac{\sqrt{n}}{\mathbb{E}|X|} \cdot \sqrt{\log K} \cdot \log n.$$

Moreover, there exists a convex polytope P_0 with K facets such that

$$\gamma^+(\partial P_0) \geq C'\frac{\sqrt{n}}{\mathbb{E}|X|}\sqrt{\log K}.$$

Here C, C' and c stand for absolute constants, independent of P, γ and n.

The upper bound part of Theorem 5.3, up to a factor of $\log n$, is a generalization of Nazarov's estimate for the standard Gaussian measure, while the lower bound was new even in the Gaussian case. For a generalization of the same result by Nazarov in an entirely different set up see Kane [48].

Additionally, in [68] I obtained the inequality:

$$\gamma_2(Q + hB_2^n) \geq \gamma_2(Q) + \frac{\sqrt{\pi}\gamma_2^+(\partial Q)^2}{8\sqrt{n}} \cdot \left(1 - e^{-\frac{\sqrt{n}}{\sqrt{\pi}\gamma_2(\partial Q)}h}\right),$$

which is an improvement of the Gaussian concentration inequality (see Talagrand [109], Ledoux, Talagrand [63]) for small values of h.

5.3. **Current work and open problems.** The first natural question that arises is the following

Problem 5.4. *Generalize Theorem 5.2 to the case of all log-concave probability measures.*

The asymptotic equality should in this case involve additional parameters. Indeed, if γ is the normalized Lebesgue measure on a cube (which is log-concave), the conclusion of Theorem 5.2 does not hold: the surface area of the unit cube is $2n$, but $\mathbb{E}|X| \approx \sqrt{n}$ and $Var|X| \approx 1$ (see the book by Brazitikos, Giannopoulos, Valettas, Vritsiou [25]).

Another very interesting question is to find Gaussian isoperimetric inequality for origin-symmetric sets in $\mathbb{R}^n$. Frank Morgan suggested that the correct statement for this question is as follows.

Problem 5.5. *For the standard Gaussian measure γ_2, prove that $\gamma_2^+(\partial K)$ is minimized when K is a symmetric strip of the same Gaussian measure as K, provided that $\gamma_2(K) \geq \frac{1}{2}$.*

Methods of solving this question may include applying Proposition 5.1 along with S-inequality of Latała and Oleszkiewicz [61].

5.4. **Prospective projects for graduate students.** It would be very interesting to get an answer to the following question.

Problem 5.6. *Solve the exact reverse isoperimetric problem for the standard Gaussian measure on the plane:* find the convex set in $\mathbb{R}^2$ with the largest Gaussian surface area.

That question arose from the discussion with Amir Livne Bar-on. The conjecture is that it is a "very thin" strip. Expressing Gaussian surface area in terms of the support function and varying the support function may be one way to approach this question.

Another good question for a student would be the following.

Problem 5.7. *Generalize the result of Theorem 5.2 to a wider class of rotation invariant measures.*

The reason to believe that this problem will have a meaningful solution is that Theorem 5.2 holds, in particular, for measures with density $C_p e^{-\frac{|y|^p}{p}}$, which are not log-concave when $p \in (0, 1)$. There should be a condition more general then log-concavity, which would include such measures.

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