On the Log-Brunn-Minkowski conjecture
(based on various joint works with Andrea Colesanti, John Hosle, Alexander Kolesnikov, Arnaud Marsiglietti, Piotr Nayar, Artem Zvavitch.)

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Notation

- Convex bodies in $\mathbb{R}^n$ denote $K, L, M$;
- Lebesgue volume in $\mathbb{R}^n$ denote $| \cdot |$ or $| \cdot |_n$;
- Recall Minkowski sum of sets $A, B \subset \mathbb{R}^n$:
  $$A + B = \{x + y : x \in A, y \in B\}.$$  
- Support function of a convex set $K$ is
  $$h_K(y) = \sup_{x \in K} \langle x, y \rangle = \|y\|_{K^o};$$
- $h_{K+L} = h_K + h_L$;
- Unit normal to $\partial K$ at $x \in \partial K$ denote $n_x$;
- $h_K(n_x) = \langle x, n_x \rangle$;
- Second fundamental form of $\partial K$ denote $\mathrm{II}$, mean curvature $H_x = \text{tr}(\mathrm{II})$.  

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On the Log-Brunn-Minkowski conjecture
The Brunn-Minkowski inequality

Log-concavity of the Lebesgue measure

\[ |\lambda K + (1 - \lambda) L| \geq |K|^{\lambda} |L|^{1-\lambda} \]

\( \frac{1}{n} \)-concavity of the Lebesgue measure

\[ |\lambda K + (1 - \lambda) L|^{\frac{1}{n}} \geq \lambda |K|^{\frac{1}{n}} + (1 - \lambda) |L|^{\frac{1}{n}} \]

The isoperimetric inequality

For all Borel-measurable sets \( K \) with \( |K| = |B_2^n| \), one has \( |\partial K| \geq |\partial B_2^n| \).

Proof

\[ |\partial K| = \liminf_{\epsilon \to 0} \frac{|K + \epsilon B_2^n| - |K|}{\epsilon} \geq \liminf_{\epsilon \to 0} \frac{\left( |K|^{\frac{1}{n}} + \epsilon |B_2^n|^{\frac{1}{n}} \right)^n - |K|}{\epsilon} = n |K|^{\frac{n-1}{n}} |B_2^n|^{\frac{1}{n}}. \]
The local version of the Brunn-Minkowski inequality

- Fix convex sets $K$ and $L$ with support functions $h_K$ and $h_L$;
- Let $\psi : \mathbb{S}^{n-1} \to \mathbb{R}$ be given by $\psi(u) = h_L(u) - h_K(u)$;
- For $t \in [0,1]$, the body $K_t = (1-t)K + tL$ has support function $h_t = h_K + t\psi$ on $\mathbb{S}^{n-1}$;
- The Brunn-Minkowski inequality
  \[ |\lambda K + (1-\lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda} \]
  implies that $\log |K_t|$ is concave;
- Let $F(t) = |K_t|$. We deduce $(\log F)'_{t=0} \leq 0$, or
  \[ F''(0)F(0) - F'(0)^2 \leq 0. \]
The local version of the Brunn-Minkowski inequality

- \( F(t) = |K_t|, \quad h_t = h_K + t\psi, \) BM implies \( F''(0)F(0) - F'(0)^2 \leq 0. \)
- Let \( f : \partial K \to \mathbb{R} \) be given by \( f(x) = \psi(n_x) = h_L(n_x) - h_K(n_x); \)
- \( F(0) = |K|; \)
- \( F'(0) = \int_{\partial K} f; \)
- \( F''(0) = \int_{\partial K} H_x f^2 - \langle \Pi^{-1}\nabla_{\partial K} f, \nabla_{\partial K} f \rangle; \)
- Brunn-Minkowski inequality implies, and follows from
  \[
  \int_{\partial K} H_x f^2 - \langle \Pi^{-1}\nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{(\int_{\partial K} f)^2}{|K|} \leq 0.
  \]
  (Colesanti 2008; Kolesnikov-Milman 2015-2018)
Abstract observation

- Take any algebra $A$ which is a vector space over $\mathbb{R}$;
- Let $Q : A \times A \to \mathbb{R}$ be any symmetric bilinear form;
- Suppose for every $a \in A$,
  \[ Q(a, a) \leq 0. \]  
  \(1\)
- Fix any element $z \in A$;
- For all $t \in \mathbb{R}$ we have $Q(a + tz, a + tz) \leq 0$, or equivalently
  \[ Q(a, a) + 2tQ(a, z) + t^2Q(z, z) \leq 0; \]  
- Optimize in $t$, plug optimal $t = -\frac{Q(a, z)}{Q(z, z)}$, get the Schwartz inequality
  \[ Q(a, a) \leq \frac{Q(a, z)^2}{Q(z, z)} \leq 0 \]  
  \(2\)
- (2) is sharper than (1);
- (2) is invariant under $a \to a + tz$. 

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On the Log-Brunn-Minkowski conjecture
Optimizing the local version of the Brunn-Minkowski inequality

- The local version of the (multiplicative) Brunn-Minkowski inequality:

\[
\int_{\partial K} H_x f^2 - \langle II^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \left( \frac{\int_{\partial K} f}{|K|} \right)^2 \leq 0.
\]

- Pick the special function \( z(x) = \langle x, n_x \rangle (= h_K(n_x)) \);
- Optimize with respect to \( f(x) + tz(x) \), using Schwartz inequality get a strengthening

\[
\int_{\partial K} H_x f^2 - \langle II^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{n-1}{n} \frac{\left( \int_{\partial K} f \right)^2}{|K|} \leq 0.
\]

- When \( K = B_2^n \), we get the sharp Poincare inequality on \( S^{n-1} \):

\[
\int_{S^{n-1}} f^2 - \left( \int_{S^{n-1}} f \right)^2 \leq \frac{1}{n-1} \int_{S^{n-1}} |\nabla_{\sigma} f|^2,
\]

where \( \int_{S^{n-1}} \) is normalized.

- The first eigenvalue of \( \Delta \) on \( S^{n-1} \) is \( n-1 \), and the above is sharp.
The Local Brunn-Minkowski inequality

\[
\int_{\partial K} H_x f^2 - \langle II^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{n-1}{n} \frac{\left(\int_{\partial K} f\right)^2}{|K|} \leq 0
\]

is invariant under \( f \to f + t \langle x, n_x \rangle \) ("times change");

- It is also invariant under \( f \to sf \) (dilating);

- Recall the definition of mixed volumes of convex bodies \( K \) and \( M \), for \( k = 1, \ldots, n \):

\[
V_k(K, M) = \frac{(n-k)!}{n!} |K + tM|_{t=0}^{(k)};
\]

- WLOG suppose that \( f(x) = h_M(n_x) \) for some convex body \( M \) (or else add a large multiple of \( h_K(n_x) \)). Get Minkowski’s second inequality:

\[
V_2(K, M) \leq \frac{V_1(K, M)^2}{|K|}.
\]

- Upshot: the Minkowski second inequality is equivalent to the Brunn-Minkowski inequality.
The $L^2$ proof of the Brunn-Minkowski inequality (Kolesnikov-Milman)

- Goal: $\int_{\partial K} H_x f^2 - \langle \Pi^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle - \frac{n-1}{n} \left( \frac{\int_{\partial K} f}{|K|} \right)^2 \leq 0$.
- Let $u : K \to \mathbb{R}$ be any function such that $\langle \nabla u, n_x \rangle = f(x)$ for $x \in \partial K$.
- By divergence theorem, $\int_{\partial K} f = \int_K \Delta u$.

**Lemma (Kolesnikov, Milman 2015)**

$$\int_{\partial K} H_x f^2 - \langle \Pi^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle \leq \int_K (\Delta u)^2 - \| \nabla^2 u \|^2.$$

- Goal follows from finding for every $f : \partial K \to \mathbb{R}$ such $u : K \to \mathbb{R}$ with $\langle \nabla u, n_x \rangle = f(x)$ and

$$\mathbb{E} \| \nabla^2 u \|^2 \geq \text{Var}(\Delta u) + \frac{1}{n} (\mathbb{E} \Delta u)^2.$$  

**Solvability of the Neumann system**

Let $\Delta u = \text{const}$, with the Neumann boundary condition $\langle \nabla u, n_x \rangle = f(x)$.

- For any symmetric matrix $A$, $\| A \|^2_{HS} \geq \frac{\text{tr}(A)^2}{n}$; thus $\| \nabla^2 u \|^2 \geq \frac{1}{n} (\Delta u)^2$.  

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On the Log-Brunn-Minkowski conjecture
The Log-Brunn-Minkowski conjecture

Logarithmic sum (Definition)

\[ \lambda K +_0 (1 - \lambda)L = \bigcap_{u \in S^{n-1}} \{ x \in \mathbb{R}^n : |\langle u, x \rangle| \leq h_K(u)^{\lambda} h_L(u)^{1-\lambda} \}. \]

Note, by AMGM, \( \lambda K +_0 (1 - \lambda)L \subset \lambda K + (1 - \lambda)L \).

Log-Brunn-Minkowski conjecture (Böröczky, Lutwak, Yang, Zhang 2011)

For origin-symmetric convex sets \( K \) and \( L \) in \( \mathbb{R}^n \),

\[ |\lambda K +_0 (1 - \lambda)L| \geq |K|^\lambda |L|^{1-\lambda}. \]

- Equivalent to uniqueness of solution of certain Monge-Ampere equations, questions go back to Firey;
- True for \( n = 2 \) (Böröczky, Lutwak, Yang, Zhang 2011), (Stancu for polytopes);
- True for unconditional sets (Saraglou 2013; Cordero-Fradelizi-Maurey; Boroczky, Kalantzopoulos 2020 – more general result);
- True for complex convex bodies (Rotem 2017).
The local version of the Log-Brunn-Minkowski inequality

- Fix convex sets $K$ and $L$ with support functions $h_K$ and $h_L$;
- Let $\psi : \mathbb{S}^{n-1} \to \mathbb{R}$ be given by $\psi(u) = \frac{h_L(u)}{h_K(u)}$;
- Locally, $K_t := tK + 0(1-t)L$ has support function
  $$h_t = h_K \psi^t = h_K + t\varphi + O(t^2),$$
  where $\varphi = h_K \log \frac{h_L}{h_K}$;
- The Log-Brunn-Minkowski inequality implies that $\log |K_t|$ is concave;
- Let $F(t) = |K_t|$. We deduce $(\log F)'_{t=0} \leq 0$, or $F''(0)F(0) - F'(0)^2 \leq 0$.
- Let $f : \partial K \to \mathbb{R}$ be given by $f(x) = \varphi(n_x) = h_K(n_x) \log \frac{h_L(n_x)}{h_K(n_x)}$;
- $F(0) = |K|$;
- $F'(0) = \int_{\partial K} f$;
- $F''(0) = \int_{\partial K} H_x f^2 - \langle \mathbb{I}^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle + \int_{\partial K} \frac{f^2}{\langle x, n_x \rangle}$. 
The local version of the Log-Brunn-Minkowski inequality

**Theorem (Colesanti, L, Marsiglietti 2016)**

The Log-Brunn-Minkowski inequality would imply, for every symmetric convex $K$ and every even function $f : \partial K \to \mathbb{R}$,

$$
\int_{\partial K} H_x f^2 - \langle \Pi^{-1} \nabla \partial K f, \nabla \partial K f \rangle + \int_{\partial K} \frac{f^2}{\langle x, nx \rangle} \leq \frac{\left( \int_{\partial K} f \right)^2}{|K|}.
$$

**Colesanti-L-Marsiglietti**

The local version of the Log-Brunn-Minkowski inequality is true when $K = B_2^n$.

- Indeed, the Local Log-Brunn-Minkowski inequality with $K = B_2^n$ is equivalent to the following Poincare inequality:

  $$
  \text{Var}_{S^{n-1}}(f) \leq \frac{1}{n} \mathbb{E}_{S^{n-1}} |\nabla \sigma f|^2,
  $$

  for all even functions $f$, which is known to be true, moreover, with constant $\frac{1}{2n} < \frac{1}{n}$. 

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**On the Log-Brunn-Minkowski conjecture**
The local version of the Log-Brunn-Minkowski inequality

Kolesnikov-Milman

The local version of the Log-Brunn-Minkowski inequality is true when $K = B_p^n$, for all $p \in [2, \infty]$.

Chen-Huang-Li-Liu; Putterman

The local version of the Log-Brunn-Minkowski inequality implies the global version of the Log-Brunn-Minkowski inequality!

- However, when $K$ is fixed, no global result follows. The global conjecture
  
  \[ |\lambda K + (1 - \lambda)L| \geq |K|^\lambda |L|^{1-\lambda} \]

  with arbitrary symmetric $L$ is not known for any $K$.

- Could one prove the Local Log BM for some nice “speed function” $f$, for all $K$? (one such answer will come after two slides...)
(Kolesnikov-Milman) The Local Log BM inequality

\[
\int_{\partial K} H_x f^2 - \langle \Pi^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle + \int_{\partial K} \frac{f^2}{\langle x, n_x \rangle} \leq \frac{\left( \int_{\partial K} f \right)^2}{|K|}
\]

is invariant under \( f \to f + t \langle x, n_x \rangle \).

(Putterman) Therefore, it is equivalent to the strengthening of Minkowski’s second inequality

\[
n(n-1) V_2(K, M) + \int_{\partial K} \frac{h_M^2(n_x)}{\langle x, n_x \rangle} dH_{n-1}(x) \leq \frac{n^2 V_1(K, M)^2}{|K|}.
\]

Furthermore, the Local (and global) Log BM is invariant under linear transformations.

In the case of Log-Brunn-Minkowski conjecture, the invariance under \( f \to f + t \langle x, n_x \rangle \) corresponds to the invariance of the global version under \( L \to tL \), while the invariance under \( f \to sf \) corresponds to “time change”.
Example: $K = B_{\infty}^n$; it was previously known to Emanuel Milman

- The inequality

$$n(n - 1)V_2(K, M) + \int_{\partial K} \frac{h_M^2(n_x)}{\langle x, n_x \rangle} dH_{n-1}(x) \leq \frac{n^2 V_1(K, M)^2}{|K|}$$

becomes (using symmetry!)

$$n(n - 1)V_2(B_{\infty}^n, M) + 2 \cdot 2^{n-1} \sum_{i=1}^{n} h_M^2(e_i) \leq 2^{-4} \cdot 4 \cdot 2^{2n-2} \left( \sum_{i=1}^{n} h_M(e_i) \right)^2.$$

- Mixed volumes are monotone, thus $V_2(B_{\infty}^n, M) \leq V_2(B_{\infty}^n, B_M)$, where $B_M$ is the parallelepiped with sides $2h_M(e_1), \ldots, 2h_M(e_n)$.

- $n(n - 1)V_2(B_{\infty}^n, B_M) = 4 \cdot 2^{n-2} \sum_{i \neq j} h_M(e_i)h_M(e_j)$.

- Thus the inequality boils down to an equality

$$\left( \sum_{i=1}^{n} h_M(e_i) \right)^2 = \left( \sum_{i=1}^{n} h_M(e_i) \right)^2.$$
The local Log-Brunn-Minkowski inequality for interval $M$

**Theorem (Kolesnikov, L, 2020+)**

For every symmetric convex bounded set $K$ in $\mathbb{R}^n$ with non-empty interior, for $f(x) = t\langle x, n_x \rangle + |\langle v, n_x \rangle|$, for any $t \in \mathbb{R}$ and any $v \in \mathbb{R}^n$, the Local Log-Brunn-Minkowski inequality is true. In other words,

$$
\int_{\partial K} H_x \langle n_x, v \rangle^2 - \langle \mathbb{I}^{-1} \nabla \partial K | \langle n_x, v \rangle |, \nabla \partial K | \langle n_x, v \rangle \rangle + \frac{\langle n_x, v \rangle^2}{\langle x, n_x \rangle} \leq
$$

$$
\frac{1}{|K|} \left( \int_{\partial K} |\langle n_x, v \rangle| \right)^2.
$$

Furthermore, the equality is attained if and only if $K = C + [-v, v]$ for some symmetric convex $C \subset w^\perp$, for some vector $w \in \mathbb{R}^n \setminus v^\perp$. 
The local Log-Brunn-Minkowski inequality for interval \( M \)

**Proof**

- Recall the support function of an interval: \( h_{[-v,v]}(u) = |\langle u, v \rangle| \);
- By invariance properties, it is enough to show that

\[
n(n-1)V_2(K, M) + \int_{\partial K} \frac{h_M^2(n_x)}{\langle x, n_x \rangle} \leq \frac{n^2 V_1(K, M)^2}{|K|}
\]

is true when \( M = [-v, v] \), for any vector \( v \in \mathbb{R}^n \);
- Cauchy’s projection formula:

\[
nV_1(K, [-v, v]) = |K + t[-v, v]|'_{t=0} = 2|K|v_{\perp} = \int_{\partial K} |\langle n_x, v \rangle|;
\]

- The function \( |K + t[-v, v]| = |K| + 2t|v| \cdot |K|v_{\perp} \) is linear in \( t \) and thus

\[
n(n-1)V_2(K, M) = |K + t[-v, v]|''_{t=0} = 0.
\]
- Our goal rewrites:

\[
\int_{S^{n-1}} \frac{|\langle u, v \rangle|^2}{h_K(u)} dS_K(u) \leq \frac{4|K|v_{\perp}^2}{|K|}.
\]
Goal: \( \int_{\mathbb{S}^{n-1}} \left( \frac{|\langle u, v \rangle|}{h_K(u)} \right) |\langle u, v \rangle| dS_K(u) \leq \frac{4|K| |v^\perp|^2}{|K|} \)

By Fubini’s theorem, for every \( u \in \mathbb{S}^{n-1} \), \(|K| = \int_{-h_K(u)}^{h_K(u)} |K \cap (u^\perp + tu)| dt\), and thus
\[
\frac{1}{h_K(u)} \leq \frac{2}{|K|} |K \cap u^\perp|.
\]

Since the projection of a subset is smaller than the projection of a set,
\[
\frac{|\langle u, v \rangle|}{h_K(u)} \leq \frac{2}{|K|} |K \cap u^\perp| \cdot |\langle u, v \rangle| = \frac{2}{|K|} |K \cap u^\perp| |v^\perp| \leq \frac{2}{|K|} |K| |v^\perp|.
\]

We conclude
\[
\int_{\mathbb{S}^{n-1}} \left( \frac{|\langle u, v \rangle|}{h_K(u)} \right) |\langle u, v \rangle| dS_K(u) \leq \frac{2|K| |v^\perp|}{|K|} \int_{\mathbb{S}^{n-1}} |\langle u, v \rangle| dS_K = \frac{4|K| |v^\perp|^2}{|K|}.
\]

(the last passage is the Cauchy’s projection formula again.) \(\square\)
When $M = [-e_1, e_1] \times [-e_2, e_2]$, the Local Log BM inequality

$$n(n-1)V_2(K, M) + \int_{\partial K} \frac{h_M^2}{\langle x, n_x \rangle} dH_{n-1}(x) \leq \frac{n^2 V_1(K, M)^2}{|K|}$$

becomes

**Conjecture: Local Log BM when $M$ is a square**

$$8|K|\text{span}(e_1, e_2)^\perp + \int_{S^{n-1}} \frac{(|u_1| + |u_2|)^2}{h_K(u)} dS_K(u) \leq \frac{4 (|K|e_1^\perp | + |K|e_2^\perp |)^2}{|K|}.$$  

- This does not reduce to the case of one interval: (Example – hexagon on the plane which is close to the square.)

**Observation**

If the above inequality is true for all $K$, then the Local Log-Brunn-Minkowski inequality holds whenever

- $K$ is any symmetric convex body and $M$ is a zonoid (limit of a sum of intervals), or
- $K$ is a zonoid and $M$ is any symmetric convex body.
Another inequality

Remark

Suppose the Log-Brunn-Minkowski conjecture holds. Then, for all symmetric convex $K$ and $M$,

$$n(n-1)V_2(K,M) + \int_{\partial K} \frac{h_M^2(n_x)}{\langle x, n_x \rangle} \leq \frac{n^2 V_1(K,M)^2}{|K|}.$$ 

Since $V_2(K,M) \geq 0$, this implies

$$\int_{\partial K} \frac{h_M^2(n_x)}{\langle x, n_x \rangle} \leq \frac{n^2 V_1(K,M)^2}{|K|}.$$ 

Equivalently,

$$\int_{\partial K} \frac{\|n_x\|^2}{\langle x, n_x \rangle} \leq \frac{1}{|K|} \left( \int_{\partial K} \|n_x\| \right)^2,$$

where $\| \cdot \| = \| \cdot \|_M^o$ is a semi-norm.

Question: Is the above inequality true?
Another inequality

**Theorem (Kolesnikov, L, 2020+)**

For any symmetric convex bounded set $K$ in $\mathbb{R}^n$ with non-empty interior, and any semi-norm $\| \cdot \|$ on $\mathbb{R}^n$, we have

$$\int_{\partial K} \frac{\|n_x\|^2}{\langle x, n_x \rangle} \leq \frac{1}{|K|} \left( \int_{\partial K} \|n_x\| \right)^2.$$ 

Furthermore, the equality occurs if and only if $\| \cdot \| = |\cdot, v|$, for some vector $v \in \mathbb{R}^n$, and $K = C + [-v, v]$ for some symmetric convex $C \subset w^\perp$, for some vector $w \in \mathbb{R}^n \setminus v^\perp$.

**Sketch of the proof**

Recall that any semi-norm there exists a set $\Omega$ such that

$$\|u\| = \sup_{v \in \Omega} |\langle u, v \rangle|.$$ 

Similarly to the previous proof, we show

$$\int_{\partial K} \sup_{v \in \Omega} |\langle n_x, v \rangle|^2 \leq \frac{1}{|K|} \left( \int_{\partial K} \sup_{v \in \Omega} |\langle n_x, v \rangle| \right)^2.$$
Theorem (Kolesnikov, L, 2020+)

For any symmetric convex bounded set $K$ in $\mathbb{R}^n$ with non-empty interior, and any semi-norm $\| \cdot \|$ on $\mathbb{R}^n$, we have

$$\int_{\partial K} \frac{\| n_x \|^2}{\langle x, n_x \rangle} \leq \frac{2C_{\text{poin}}(K)}{\text{inrad}(K)} \cdot \frac{1}{|K|} \left( \int_{\partial K} \| n_x \| \right)^2,$$

where $\text{inrad}(K)$ is the radius of the largest ball inside $K$, and $C_{\text{poin}}(K) = \inf_{v : 1-Lip} \sqrt{\text{Var}_K(v)}$.

Note that $\frac{2C_{\text{poin}}(K)}{\text{inrad}(K)} < 1$ e.g. for $K = B_2^n$, in which case this estimate beats the previous estimate.
Proof

Lemma

Let $K$ be $C^2$-smooth strictly convex body in $\mathbb{R}^n$. Let $\| \cdot \|$ be an arbitrary semi-norm in $\mathbb{R}^n$. Let $u : K \rightarrow \mathbb{R}$ be any $C^2$ function such that $\langle \nabla u, n_x \rangle = \| n_x \|$ for all $x \in \partial K$. Then

$$\int_K \| \nabla^2 u \|^2_{HS} \leq \int_K (\Delta u)^2.$$

Proof of the Lemma

- Recall (Kolesnikov, Milman): when $\langle \nabla u, n_x \rangle = f : \partial K \rightarrow \mathbb{R}$,

$$\int_{\partial K} H_x f^2 - \langle \Pi^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle \leq \int_K (\Delta u)^2 - \| \nabla^2 u \|^2.$$

- When $f(x) = \| n_x \| = \| n_x \|_{\mathcal{M}^0}$, we have

$$\int_{\partial K} H_x f^2 - \langle \Pi^{-1} \nabla_{\partial K} f, \nabla_{\partial K} f \rangle = |K + tM|''_0 = n(n-1)V_2(K, M) \geq 0. \square$$
Proof

- Let $u : K \rightarrow \mathbb{R}$ be the solution of the Neumann system

\[
\langle \nabla u, n_x \rangle = \|n_x\|, \quad x \in \partial K,
\]

and

\[
\Delta u = \frac{\int_{\partial K} \|n_x\|}{|K|}.
\]

- \[
\int_{\partial K} \frac{\|n_x\|^2}{\langle x, n_x \rangle} \leq \frac{1}{r} \int_{\partial K} |\nabla u| \langle \nabla u, n_x \rangle, \quad \text{where } r = \operatorname{inrad}(K) = \min\langle x, n_x \rangle \text{ and we used that } \langle \nabla u, n_x \rangle \geq 0.
\]

- For any $\alpha, \beta > 0$,

\[
div(\|\nabla u\| \nabla u) = \Delta u \|\nabla u\| + \langle \nabla^2 u \frac{\nabla u}{\|\nabla u\|}, \nabla u \rangle \leq \]

\[
\frac{\alpha}{2} (\Delta u)^2 + \frac{1}{2\alpha} \|\nabla u\|^2 + \frac{\beta}{2} \|\nabla^2 u\|_{HS}^2 + \frac{1}{2\beta} \|\nabla u\|^2.
\]
Proof (continued)

Thus, by divergence theorem, we get

\[
\int_{\partial K} \left\langle x, n_x \right\rangle \leq \frac{1}{r} \int_K \frac{\alpha}{2} (\Delta u)^2 + \frac{1}{2\alpha} |\nabla u|^2 + \frac{\beta}{2} \|\nabla^2 u\|_{HS}^2 + \frac{1}{2\beta} |\nabla u|^2 \, dx \leq \\
\frac{1}{r} \int_K \frac{\alpha}{2} (\Delta u)^2 + \left( \frac{C_{poin}^2}{2\alpha} + \frac{\beta}{2} + \frac{C_{poin}^2}{2\beta} \right) \|\nabla^2 u\|_{HS}^2,
\]

where in the last line we used the Poincare inequality coordinate-wise for \(\nabla u\), in view of the fact that \(u\) is even and thus \(\int_K \nabla u = 0\).

We let \(\alpha = \beta = C_{poin}\), and use the Lemma

\[
\int_K \|\nabla^2 u\|_{HS}^2 \leq \int_K (\Delta u)^2,
\]

in order to conclude

\[
\int_{\partial K} \left\langle x, n_x \right\rangle \leq \frac{2C_{poin}}{r} \cdot \int_K (\Delta u)^2.
\]
Proof (continued)

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Proof

- It remains to recall that $\Delta u$ is a constant function, and thus

$$\int_K (\Delta u)^2 \, dx = \frac{\left( \int_K \Delta u \right)^2}{|K|} = \frac{\left( \int_{\partial K} \| n_x \| \right)^2}{|K|},$$

where in the last passage, the Divergence Theorem was used.

- We conclude that

$$\int_{\partial K} \frac{\| n_x \|^2}{\langle x, n_x \rangle} \leq \frac{2C_{po} \cdot \left( \int_{\partial K} \| n_x \| \right)^2}{r \cdot |K|},$$

and the theorem follows. □
The $L_p$-Brunn-Minkowski conjecture

$L_p$-Minkowski sum (Definition)

$$\lambda K +_p (1 - \lambda) L = \bigcap_{u \in S^{n-1}} \{ x \in \mathbb{R}^n : |\langle u, x \rangle|^p \leq \lambda h_K(u)^p + (1 - \lambda) h_L(u)^p \}.$$ 

$L_p$-Brunn-Minkowski conjecture (Böröczky, Lutwak, Yang, Zhang 2011)

For origin-symmetric convex sets $K$ and $L$ in $\mathbb{R}^n$, for $p \in [0, 1]$

$$|\lambda K +_p (1 - \lambda) L| \geq |K|^\lambda |L|^{1-\lambda}.$$ 

Equivalently, by homogeneity (and/or the earlier story)

$$|\lambda K +_p (1 - \lambda) L|^{\frac{p}{n}} \geq \lambda |K|^{\frac{p}{n}} + (1 - \lambda) |L|^{\frac{p}{n}}.$$ 

- For $p \in [0, 1]$,

$$\lambda K +_0 (1 - \lambda) L \subset \lambda K +_p (1 - \lambda) L \subset \lambda K + (1 - \lambda) L.$$ 

- The conjecture interpolates between the Log-Brunn-Minkowski conjecture ($p = 0$) and the Brunn-Minkowski inequality ($p = 1$).
Kolesnikov-Milman developed the local version of the $L_p$-Brunn-Minkowski inequality

$$n(n - 1)V_2(K, M) + (1 - p) \int_{\partial K} \frac{h_M^2}{\langle x, n_x \rangle} dH_{n-1}(x) \leq \frac{n - p}{n} \frac{n^2 V_1(K, M)^2}{|K|}.$$  

Kolesnikov-Milman: true for $p \in [1 - cn^{-1.5}, 1]$!

Chen-Huang-Li-Liu: local implies global (with equality cases)

Puttermann: local implies global (simple and useful proof)

Conclusion: the $L_p$-Brunn-Minkowski conjecture

$$|\lambda K + p (1 - \lambda)L|^{\frac{p}{n}} \geq \lambda |K|^{\frac{p}{n}} + (1 - \lambda)|L|^{\frac{p}{n}}.$$  

is true when $p \in [1 - cn^{-1.5}, 1]$!
The $L_p$-Brunn-Minkowski conjecture

**Theorem (Hosle, Kolesnikov, L 2020+)**

For origin-symmetric convex sets $K$ and $L$ in $\mathbb{R}^n$ such that $K \subset L$, for $p \in [1 - cn^{-0.75}, 1]$

$$|\lambda K + p(1 - \lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda}.$$ 

Remark: note that this is not the dilation-invariant version.
Log-concave measures: preliminaries

Log-concave functions

A function is called log-concave if its logarithm is concave, i.e.
\[ f(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda f(y)^{1-\lambda}. \]

Log-concave measures

A measure \( \mu \) is called log-concave if
\[ \mu(\lambda K + (1 - \lambda)L) \geq \mu(K)^\lambda \mu(L)^{1-\lambda}. \]

Borell’s theorem (which implies Brunn-Minkowski)

A measure with log-concave density is log-concave.

- Gaussian measure \( \gamma \) with density \( \frac{1}{\sqrt{2\pi}^n} e^{-\frac{|x|^2}{2}} \);
- Lebesgue measure;
- Poisson density...
The $L_p$-Brunn-Minkowski conjecture for measures

**Theorem (Saraglou 2014)**

If the $L_p$-Brunn-Minkowski conjecture holds for some $p \in [0,1]$, then for any even log-concave measure $\mu$ and any pair of origin-symmetric convex sets $K$ and $L$ in $\mathbb{R}^n$,

$$
\mu(\lambda K + p(1 - \lambda)L) \geq \mu(K)^\lambda \mu(L)^{1-\lambda}.
$$

- Considering the case $p = 0$ and $K = aL$, note that the above would imply the B-conjecture of Banazchyk-Latala, posed in the 1990s.
- Cordero-Fradelizi-Maurey 2008: true when $K = aL$ and $\mu$ is Gaussian.
- In the absence of homogeneity, the inequality no longer improves to an additive version...
The $L_p$-Brunn-Minkowski conjecture for measures

Theorem (Saroglou 2014)

If the $L_p$-Brunn-Minkowski conjecture holds for some $p \in [0,1]$, then for any even log-concave measure $\mu$ and any pair of origin-symmetric convex sets $K$ and $L$ in $\mathbb{R}^n$,

$$\mu(\lambda K + p (1 - \lambda)L) \geq \mu(K)^{\lambda} \mu(L)^{1-\lambda}.$$ 

- Considering the case $p = 0$ and $K = aL$, note that the above would imply the B-conjecture of Banazchyk-Latala, posed in the 1990s.
- Cordero-Fradelizi-Maurey 2008: true when $K = aL$ and $\mu$ is Gaussian.
- In the absence of homogeneity, the inequality no longer improves to an additive version... Except it does!

L, Marsiglietti, Nayar, Zvavitch 2017

If the Log-Brunn-Minkowski conjecture holds, then for every even log-concave measure $\mu$ and any pair of origin-symmetric convex sets $K$ and $L$ in $\mathbb{R}^n$,

$$\mu\left(\lambda K + \left(1 - \lambda\right)L\right)^{\frac{1}{n}} \geq \lambda \mu(K)^{\frac{1}{n}} + \left(1 - \lambda\right)\mu(L)^{\frac{1}{n}}.$$
The dimensional Brunn-Minkowski

**Theorem (Hosle, Kolesnikov, L 2020+)**

If the Log-Brunn-Minkowski conjecture holds, then for every even log-concave measure \( \mu \) and any pair of origin-symmetric convex sets \( K \) and \( L \) in \( \mathbb{R}^n \),

\[
\mu(\lambda K + p (1 - \lambda)L)^\frac{p}{n} \geq \lambda \mu(K)^\frac{p}{n} + (1 - \lambda) \mu(L)^\frac{p}{n}.
\]

(3)

Moreover, (3) strengthens when \( p \) decreases.

**Conjecture (Gardner, Zvavitch 2007)**

For an even log-concave measure \( \mu \), and symmetric convex sets \( K \) and \( L \),

\[
\mu(\lambda K + (1 - \lambda)L)^\frac{1}{n} \geq \lambda \mu(K)^\frac{1}{n} + (1 - \lambda) \mu(L)^\frac{1}{n}.
\]

• Tkocz-Nayar: the symmetry assumption cannot be replaced by simply origin in the interior, even in the Gaussian case.

**Theorem (Kolesnikov, L 2018)**

For the Gaussian measure \( \mu \), and convex sets \( K \) and \( L \) containing the origin,

\[
\mu(\lambda K + (1 - \lambda)L)^\frac{1}{2n} \geq \lambda \mu(K)^\frac{1}{2n} + (1 - \lambda) \mu(L)^\frac{1}{2n}.
\]
The dimensional Brunn-Minkowski

- Eskenazis-Moschidis: for the Gaussian measure $\gamma$ and for symmetric convex sets $K$ and $L$,

$$
\gamma(\lambda K + (1 - \lambda)L)^{\frac{1}{n}} \geq \lambda \gamma(K)^{\frac{1}{n}} + (1 - \lambda)\gamma(L)^{\frac{1}{n}}.
$$

**Theorem (Kolesnikov, L 2020+)**

Fix $a \in [0, 1]$. For the Gaussian measure $\gamma$ and for symmetric convex sets $K$ and $L$ with $\gamma(K) \geq a$, $\gamma(L) \geq a$,

$$
\gamma(\lambda K + (1 - \lambda)L)^{\frac{1}{n-F(a)}} \geq \lambda \gamma(K)^{\frac{1}{n-F(a)}} + (1 - \lambda)\gamma(L)^{\frac{1}{n-F(a)}},
$$

where $\frac{1}{n-F(a)} \rightarrow a \rightarrow 1 \infty$.

**Remarks**

- The power in this inequality tends to infinity even in the non-symmetric case, as is implied by Ehrhard’s inequality (but the above is not related to Ehrhard’s inequality).

- The function $F(a) := \frac{1}{a} J_{n+1} \circ J_{n-1}^{-1}(a)$, where $J_p(R) := \int_0^R t^p e^{-\frac{t^2}{2}} \, dt$, and the rate in the previous theorem is optimal up to a dimensional constant.
Moreover,

**Theorem (Kolesnikov, L 2020+)**

Fix $a \in [0, 1]$. Let $\mu$ be a log-concave measure with twice-differentiable density $e^{-V}$, and suppose $\nabla^2 V$ is uniformly strictly non-singular everywhere. Then for symmetric convex sets $K$ and $L$ with $\mu(K) \geq a$, $\mu(L) \geq a$,

$$\mu(\lambda K + (1 - \lambda)L)^{p(a)} \geq \lambda \mu(K)^{p(a)} + (1 - \lambda)\mu(L)^{p(a)},$$

where $p(a) \to a \to 1 \infty$.

Additionally,

**Theorem (Kolesnikov, L 2020+)**

Let $\mu$ be the measure with density $C_n e^{-\|x\|_1}$. Then for symmetric convex sets $K$ and $L$,

$$\mu(\lambda K + (1 - \lambda)L)^{\frac{c}{n \log n}} \geq \lambda \mu(K)^{\frac{c}{n \log n}} + (1 - \lambda)\mu(L)^{\frac{c}{n \log n}}.$$
The mixed $L_p$-Brunn-Minkowski and dimensional conjecture for measures

**Theorem (Hosle, Kolesnikov, L 2020+)**

Let $\gamma$ be the Gaussian measure, and let $K$ and $L$ be symmetric convex sets containing the ball $rB_2^n$. Then for any $\lambda > 0$,

1. $\gamma(\lambda K + p(1 - \lambda)L) \geq \gamma(K)^\lambda \gamma(L)^{1 - \lambda}$, whenever $p \geq 0$ and
   $$p \geq 1 - \frac{2r^2}{n+1}.$$

2. In particular, the Gaussian Log-Brunn-Minkowski inequality holds for all convex sets $K$ and $L$ containing $\sqrt{0.5(n+1)}B_2^n$.

3. More generally, $\gamma(\lambda K + p(1 - \lambda)L)^\frac{q}{n} \geq \lambda \gamma(K)^\frac{q}{n} + (1 - \lambda) \gamma(L)^\frac{q}{n}$, provided that
   $$4q + \frac{n+1}{r^2} (1 - p) \leq 2.$$

4. Assuming further that $K \subseteq L$, we show that
   $$\gamma(\lambda K + p(1 - \lambda)L) \geq \gamma(K)^\lambda \gamma(L)^{1 - \lambda},$$
   whenever $p \geq 0$ and $p \geq 1 - \frac{r}{\sqrt{n+0.25}}$.

- In one of the steps of the proof, we deduced the “local to global” result for general measures, following the approach of Putterman.
Thanks for your attention!