HOME WORK 4: INVARIANTS, GAMES, ALGEBRA

1. INVARIANTS AND GAMES

1. An ordered triple of numbers is given. It is permitted to perform the following operation on the triple: to change two of them, say a and b, to \( \frac{a+b}{\sqrt{2}} \) and \( \frac{a-b}{\sqrt{2}} \). Is it possible to obtain the triple \((1, \sqrt{2}, 1 + \sqrt{2})\) from the triple \((2, \sqrt{2}, \frac{1}{\sqrt{2}})\) using this operation?

2. In Determinant Tic-Tac-Toe, Player 1 enters a 1 in an empty \(3 \times 3\) matrix. Player 0 counters with a 0 in a vacant position, and play continues in turn until the \(3 \times 3\) matrix is completed with five 1s and four 0s. Player 0 wins if the determinant is 0 and player 1 wins otherwise. Assuming both players pursue optimal strategies, who will win and how?

3. Let \(n \geq 2\) be an integer and \(T_n\) be the number of nonempty subsets \(S\) of \(\{1, 2, 3, ..., n\}\) with the property that the average of the elements of \(S\) is an integer. Prove that \(T_n - n\) is always even.

2. ALGEBRA

1. Show that for no positive integer \(n\) can both \(n + 3\) and \(n^2 + 3n + 3\) be perfect cubes.

2. Let \(a\) and \(b\) be coprime integers greater than 1. Prove that for no \(n \geq 0\) is \(a^{2n} + b^{2n}\) divisible by \(a + b\).

3. Prove that any integer can be written as the sum of five perfect cubes.

4. Solve in real numbers the equation
\[
\sqrt[3]{x-1} + \sqrt[3]{x} + \sqrt[3]{x+1} = 0.
\]