HOME WORK 4, PUTNAM PREPARATION

1. Find the largest real number k with the property that for all fourth-degree polynomials $P(x) = x^4 + ax^3 + bx^2 + cx + d$ whose zeros are all real and positive, one has $(b - a - c)^2 \geq kd$, and determine when equality holds.

2. Find all polynomials whose coefficients are equal either to 1 or -1 and whose zeros are all real.

3. Prove that there are unique positive integers a, n such that $a^{n+1} - (a+1)^n = 2001$.

4. Find all polynomials $P(x)$ with integer coefficients satisfying $P(P'(x)) = P'(P(x))$ for all $x \in \mathbb{R}$.

5. Let $P(x)$ be a polynomial of degree $n > 3$ whose zeros $x_1 < x_2 < x_3 < ... < x_{n-1} < x_n$ are real. Prove that

$$P'(\frac{x_1 + x_2}{2})P'(\frac{x_{n-1} + x_n}{2}) \neq 0.$$

6. Do there exist polynomials $a(x)$, $b(x)$, $c(x)$, $d(x)$ such that the equality

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)$$

holds identically?

7. Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + ... + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers $c_n r, c_n r^2 + c_{n-1} r, c_n r^3 + c_{n-1} r^2 + c_{n-2} r, ..., c_n r^n + ... + c_1 r$ are integers.