HW 8: ANALYSIS

1. The sequence $a_0, a_1, a_2, \ldots$ satisfies
   
   $$a_{m+n} + a_{m-n} = \frac{a_{2m} + a_{2n}}{2}$$

   for all nonnegative integers $m$ and $n$ with $m \geq n$. If $a_1 = 1$, determine $a_n$.

2. Compute
   
   $$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \ldots}}}}$$

3. Show that the series
   
   $$\frac{1}{1+x} + \frac{2}{1+x^2} + \frac{4}{1+x^4} + \frac{8}{1+x^8} + \ldots + \frac{2^n}{1+x^{2n}} + \ldots$$

   converges when $|x| > 1$, and in this case find its sum.

4. Let $a_n = \sqrt{1 + (1 + \frac{1}{n})^2} + \sqrt{1 + (1 - \frac{1}{n})^2}$, $n \geq 1$. Prove that
   
   $$\frac{1}{a_1} + \ldots + \frac{1}{a_{20}}$$

   is an integer.

5. Let $a$ and $b$ be real numbers in the interval $(0, \frac{1}{2})$ and let $f$ be a continuous real-valued function such that
   
   $$f(f(x)) = af(x) + bx,$$

   for all $x \in \mathbb{R}$. Prove that $f(0) = 0$.

6. Suppose that $f : [0, 1] \to \mathbb{R}$ has a continuous derivative and that $\int_0^1 f(x) \, dx = 0$. Prove that for every $\alpha \in (0, 1)$:
   
   $$\int_0^\alpha f(x) \, dx \leq \frac{1}{8} \max_{x \in [0,1]} |f'(x)|.$$
7. Let \( f \) be a real function on the real line with continuous third derivative. Prove that there exists a point \( a \) such that
\[
f(a)f'(a)f''(a)f'''(a) \geq 0.
\]