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# Maximal surface area of a convex set in $\mathbb{R}^n$ with respect to exponential rotation invariant measures[☆]

Galyna Livshyts

*Department of Mathematics, Kent State University, Kent, OH 44242, USA*

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ABSTRACT

Let $p$ be a positive number. Consider the probability measure $\gamma_p$ with density $\varphi_p(y) = c_{n,p}e^{-\frac{|y|^p}{p}}$. We show that the maximal surface area of a convex body in $\mathbb{R}^n$ with respect to $\gamma_p$ is asymptotically equivalent to $C(p)n^{\frac{3}{4}-\frac{1}{p}}$, where the constant $C(p)$ depends on $p$ only. This is a generalization of results due to Ball (1993) [1] and Nazarov (2003) [9] in the case of the standard Gaussian measure $\gamma_2$.



## 1. Introduction

As usual, $|\cdot|$ denotes the norm in Euclidean $n$-space $\mathbb{R}^n$, and $|A|$ stands for the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$. We will write $B_2^n = \{x \in \mathbb{R}^n : |x| \le 1\}$ for the unit ball in $\mathbb{R}^n$ and $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ for the unit $n$-dimensional sphere. We will set $\nu_n = |B_2^n| = \pi^{\frac{n}{2}}/\Gamma\left(\frac{n}{2}+1\right)$. A convex body is a compact convex set with nonempty interior.

In this paper we will study the geometric properties of measures $\gamma_p$ on $\mathbb{R}^n$ with density

$$\varphi_p(y) = c_{n,p}e^{-\frac{|y|^p}{p}},$$

where $p \in (0,\infty)$ and $c_{n,p}$ is the normalizing constant, chosen in such a way that the measure $\gamma_p$ is a probability measure.

Many interesting results are known for the case $p = 2$, where $\gamma_p$ becomes the standard Gaussian measure. One should mention the Gaussian isoperimetric inequality of Borell [3] and Sudakov and Tsirel'son [10]: fix some $a \in (0, 1)$ and $\varepsilon > 0$; then among all measurable sets $A \subset \mathbb{R}^n$ with $\gamma_2(A) = a$, the set for which $\gamma_2(A + \varepsilon B_2^n)$ has the smallest Gaussian measure is the half-space. We refer the reader to the books [2,7] for more properties of the Gaussian measure and inequalities of this type.

Mushtari and Kwapien asked a question in the reverse direction to the isoperimetric inequality: How large can the Gaussian surface area of a convex set $A \subset \mathbb{R}^n$ be? In [1] it was shown that the Gaussian surface area of a convex body in $\mathbb{R}^n$ is asymptotically bounded from above by $Cn^{\frac{1}{4}}$ as $n$ tends to infinity, where $C$ is an absolute constant. Nazarov [9] gave the complete solution to this problem by proving the sharpness of Ball's result:

$$0.28n^{\frac{1}{4}} \le \sup \gamma_2(\partial Q) \le 0.64n^{\frac{1}{4}},$$

where the supremum is taken over all $n$-dimensional convex sets. Further estimates for $\gamma_2(\partial Q)$ were very recently proved by Kane in [6].

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*E-mail address:* glivshyt@kent.edu.

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We should note that isoperimetric inequalities for rotation invariant measures were studied by Sudakov and Tsirel'son [10]. For a positive convex function $h(t)$, let $\gamma$ be the measure with density $e^{-h(\log|x|)}$. For $a > 0$, set $M_Q(a) = \gamma(aQ)$. In [10] it was proved that $M_Q'(1)$ exists whenever $Q$ is a convex body, and the minimum of $M_Q'(1)$ over all convex bodies is attained when $Q$ is a half-space. Of course, their result can be applied to the measure $\gamma_p$ by setting $h(t) = \frac{e^{pt}}{p}$. Some interesting results for manifolds with density were also provided by Bray and Morgan [4] and further generalized by Maurmann and Morgan [8].

The main goal of this paper is to complement the study of the isoperimetric problem for rotation invariant measures and prove an inverse isoperimetric inequality for $\gamma_p$. This is done using a generalization of Nazarov's method from [9].

We recall that the surface area of a convex body $Q$ with respect to the measure $\gamma_p$ is defined to be

$$\gamma_p(\partial Q) = \liminf_{\epsilon\to+0} \frac{\gamma_p((Q+\epsilon B_2^n)\setminus Q)}{\epsilon}. \tag{1}$$

One can also provide an integral formula for $\gamma_p(\partial Q)$:

$$\gamma_p(\partial Q) = \int_{\partial Q} \varphi_p(y)d\sigma(y) = c_{n,p}\int_{\partial Q} e^{-\frac{|y|^p}{p}}d\sigma(y), \tag{2}$$

where $d\sigma(y)$ stands for Lebesgue surface measure. We refer the reader to [6] for the proof for the case $p = 2$, and the proof given there readily generalizes to other $p$.

The following theorem is the main result of this paper:

**Theorem 1.** *For any positive p,*

$$c(p)n^{\frac{3}{4}-\frac{1}{p}} \le \sup \gamma_p(\partial Q) \le C(p)n^{\frac{3}{4}-\frac{1}{p}},$$

*where $C(p)$ and $c(p)$ are positive constants depending on p only. The supremum is taken over all n-dimensional convex sets.*

In this paper we will denote an asymptotic equality by "$\approx$" and an asymptotic inequality by "$\gtrsim$". Namely, $A(n) \gtrsim B(n)$ iff $A \ge B(1+o(1))$, where $o(1)$ is an infinitely small number, while $n$ (or some parameter in the context) tends to infinity. Similarly, $A(n) \approx B(n)$ iff $A(n) \gtrsim B(n) \gtrsim A(n)$. Throughout the paper, $c_1, c_2, C, C', C'', \ldots$ denote absolute constants, independent of $n$ and $p$, whose value may change from line to line; $C(p)$, $c(p)$ denote constants depending on the argument $p$ only.

Using the trick from [1, p. 413, Proposition 1] one can easily derive an estimate $\gamma_p(\partial Q) \le e^{\frac{1}{p}-1}n^{1-\frac{1}{p}}$ for any convex set. The calculation is given in Section 2, as well as some other important preliminary facts. The upper bound from Theorem 1 is obtained in Section 3, and the lower bound is shown in Section 4.

## 2. Preliminary lemmas

We recall that $\gamma_p$ is a probability measure on $\mathbb{R}^n$ with density $\varphi_p(y) = c_{n,p}e^{-\frac{|y|^p}{p}}$, where $p \in (0,\infty)$. The normalizing constant $c_{n,p}$ equals $[n\nu_n J_{n-1,p}]^{-1}$, where

$$J_{a,p} = \int_0^\infty t^a e^{-\frac{t^p}{p}}dt. \tag{3}$$

We will use an asymptotic estimate for $J_{a,p}$ (see Lemma 3 below). This estimate follows immediately from the well-known asymptotic formulas for the Gamma function, which can be obtained by the Laplace method (see, for example, [5]). In this paper we will provide several calculations in the spirit of this technique. For the sake of completeness, we shall present a short overview of the method here:

**Lemma 1.** *Let $h(x) \in C^2([a,b])$, where a and b may be infinities, and $0 \in (a,b)$. Let 0 be the global maximum point of $h(x)$ and assume for convenience that $h(0) = 0$. Assume that for any $\delta > 0$ there exists $\eta(\delta) > 0$ s.t. $h(x) < -\eta(\delta)$ for all $x \notin [-\delta,\delta]$. Assume also that $h''(0) < 0$ and that the integral $\int_a^b e^{h(x)}dx < \infty$. Then*

$$\int_a^b e^{th(x)}dx \approx \sqrt{-\frac{2\pi}{h''(0)t}}, \quad t\to\infty.$$

**Proof.** First, by the conditions of the lemma and the Taylor formula, for a sufficiently small $\epsilon > 0$ there exists a positive $\delta = \delta(\epsilon)$ such that for any $x \in (-\delta,\delta)$ it holds that

$$\left|h(x) - \frac{h''(0)x^2}{2}\right| \le \frac{\epsilon x^2}{2}.$$

Thus the integral

$$\int_{-\delta}^{\delta} e^{th(x)}dx \le \frac{1}{\sqrt{-(h''(0)+\epsilon)}}\int_{-\delta\sqrt{-(h''(0)+\epsilon)}}^{\delta\sqrt{-(h''(0)+\epsilon)}} e^{-\frac{ty^2}{2}}dy$$
$$\le \sqrt{-\frac{2\pi}{(h''(0)+\epsilon)t}}. \tag{4}$$

Note that for any constant $C>0$,

$$\int_C^{\infty} e^{\frac{-ty^2}{2}}dy \le e^{\frac{-(t-1)C^2}{2}}\int_C^{\infty} e^{-\frac{y^2}{2}}dy = C'e^{-C''t}, \tag{5}$$

and thus, as $t\to\infty$,

$$\int_{-\delta}^{\delta} e^{th(x)}dx \ge \frac{1}{\sqrt{-(h''(0)-\epsilon)}}\int_{-\delta\sqrt{-(h''(0)-\epsilon)}}^{\delta\sqrt{-(h''(0)-\epsilon)}} e^{-\frac{ty^2}{2}}dy$$
$$\gtrsim \sqrt{-\frac{2\pi}{(h''(0)-\epsilon)t}}. \tag{6}$$

It remains to show that $\int_a^b e^{th(x)}dx$ can be asymptotically estimated by the integral over the small interval about zero. Indeed, for an arbitrary $\epsilon$ we have chosen $\delta(\epsilon)$, and now by the condition of the lemma, we can pick $\eta(\delta)=\eta(\epsilon)$, so

$$\int_{(a,-\delta)\cup(\delta,b)} e^{th(x)} \le e^{-(t-1)\eta(\delta)}\int_a^b e^{h(x)}dx = C'e^{-C''t}. \tag{7}$$

Thus, by (4) and (6),

$$\sqrt{-\frac{2\pi}{(h''(0)-\epsilon)t}} \gtrsim \int_a^b e^{th(x)}dx \gtrsim \sqrt{-\frac{2\pi}{(h''(0)+\epsilon)t}}.$$

Taking $\epsilon$ small enough we finish the proof. □

We will now apply Laplace's method to deduce the asymptotic estimate for $J_{a,p}$.

**Lemma 2.** *Let $p>0$. Then*

$$J_{a,p} \approx \sqrt{\frac{2\pi}{p}}a^{\frac{1}{p}-\frac{1}{2}}a^{\frac{a}{p}}e^{-\frac{a}{p}}, \quad \text{as } a\to\infty.$$

**Proof.** We notice that

$$\int_0^{\infty} t^a e^{-\frac{t^p}{p}}dt = a^{\frac{a}{p}}e^{-\frac{a}{p}}\int_0^{\infty} e^{\frac{a}{p}\left(\log\frac{t^p}{a}-\frac{t^p}{a}+1\right)}dt = a^{\frac{a}{p}}e^{-\frac{a}{p}}a^{\frac{1}{p}}\int_0^{\infty} e^{\frac{a}{p}h(x)}dx,$$

where $h(x)=p\log x - x^p + 1$.

It is easy to check that all the conditions of Lemma 1 are satisfied:

$$h(1)=h'(1)=0; \qquad h''(1)<0 \quad \text{and} \quad \int_0^{\infty} e^{h(x)}dx<\infty.$$

In addition, for any $\delta>0$, there exists $\eta(\delta)$ such that $h(x)<-\eta(\delta)$ for all $x\notin[1-\delta,1+\delta]$. We apply Lemma 1 to finish the proof. □

Next, we would like to show that the integral

$$\int_{(1+C)a^{\frac{1}{p}}}^{\infty} t^a e^{-\frac{t^p}{p}}dt$$

is an exponentially small function as $a\to\infty$ for any absolute positive constant $C$. Indeed,

$$\int_{(1+C)a^{\frac{1}{p}}}^{\infty} t^a e^{-\frac{t^p}{p}}dt = a^{\frac{a}{p}}e^{-\frac{a}{p}}a^{\frac{1}{p}}\int_{1+C}^{\infty} e^{\frac{a}{p}h(x)}dx,$$

where $h(x) = p\log x - x^p + 1$. We note that for any $x > 1 + C$, $h(x) < h(1 + C) \le -C'$ for some positive constant $C'$. Thus, the above integral can be estimated with

$$a^{\frac{a}{p}} e^{-\frac{a}{p}} a^{\frac{1}{p}} e^{-C'\left(\frac{a}{p}-1\right)} \int_{1+C}^{\infty} e^{h(x)} dx. \tag{8}$$

Applying Lemma 2 together with (8) and the fact that $\int_0^\infty e^{h(x)} dx$ converges to some constant, we obtain the following fact:

$$\int_{(1+C)a^{\frac{1}{p}}}^{\infty} t^a e^{-\frac{t^p}{p}} dt \le C_1(p) J_{a,p} e^{-C_2(p)a}. \tag{9}$$

Next, we shall observe that the surface area is mostly concentrated in a narrow annulus. Define $\Delta_p = 1 - (2e)^{-\frac{1}{p}}$. Let

$$A_p = (1 + \Delta_p)(n-1)^{\frac{1}{p}} B_2^n \setminus (1 - \Delta_p)(n-1)^{\frac{1}{p}} B_2^n;$$

we shall call $A_p$ the concentration annulus.

**Lemma 3.** *There exist positive constants $C'(p)$ and $C''(p)$, such that $\gamma_p(\partial Q \cap A_p^c) \le C'(p) e^{-C''(p)n}$ for any convex body $Q \subset \mathbb{R}^n$.*

**Proof.** Let $Q' = Q \cap (1 - \Delta_p)(n-1)^{\frac{1}{p}} B_2^n$. Then,

$$\begin{aligned}\gamma_p(\partial Q') &= \frac{1}{n\nu_n J_{n-1,p}} \int_{\partial Q'} e^{-\frac{|y|^p}{p}} d\sigma(y) \le \frac{|\partial Q'|}{n\nu_n J_{n-1,p}} \\ &\le \frac{(1-\Delta_p)^{n-1}(n-1)^{\frac{n-1}{p}} n\nu_n}{n\nu_n J_{n-1,p}} = \frac{(1-\Delta_p)^{n-1}(n-1)^{\frac{n-1}{p}}}{J_{n-1,p}}\end{aligned}$$

which is exponentially small by the choice of $\Delta_p$.

Define now the surface $M = \partial Q \setminus (1 + \Delta_p)(n-1)^{\frac{1}{p}} B_2^n$. We can estimate $\gamma_p(M)$ using a trick from [1]. Notice that

$$e^{-\frac{|y|^p}{p}} = \int_{|y|}^{\infty} t^{p-1} e^{-\frac{t^p}{p}} dt = \int_0^\infty t^{p-1} e^{-\frac{t^p}{p}} \chi_{[-t,t]}(|y|) dt, \quad \forall y \in \mathbb{R}^n.$$

Consider $y \in M$ and $t \in \left[0, (1 + \Delta_p)(n-1)^{\frac{1}{p}}\right]$; then $\chi_{[-t,t]}(|y|) = 0$ and

$$e^{-\frac{|y|^p}{p}} = \int_{(1+\Delta_p)(n-1)^{\frac{1}{p}}}^{\infty} t^{p-1} e^{-\frac{t^p}{p}} \chi_{[-t,t]}(|y|) dt.$$

Thus

$$\begin{aligned}\gamma_p(M) &= \frac{1}{n\nu_n J_{n-1,p}} \int_M e^{-\frac{|y^p|}{p}} d\sigma(y) \\ &= \frac{1}{n\nu_n J_{n-1,p}} \int_M \int_{(1+\Delta_p)(n-1)^{\frac{1}{p}}}^{\infty} t^{p-1} e^{-\frac{t^p}{p}} \chi_{[-t,t]}(|y|) dt d\sigma(y) \\ &= \frac{1}{n\nu_n J_{n-1,p}} \int_{(1+\Delta_p)(n-1)^{\frac{1}{p}}}^{\infty} t^{p-1} e^{-\frac{t^p}{p}} |M \cap tB_2^n| dt \\ &\le \frac{1}{J_{n-1,p}} \int_{(1+\Delta_p)(n-1)^{\frac{1}{p}}}^{\infty} t^{n+p-2} e^{-\frac{t^p}{p}} dt \\ &\le C'(p) e^{-C''(p)n},\end{aligned}$$

where the last inequality follows from (9). This finishes the proof. □

**Remark 1.** Lemma 3 implies that when obtaining any polynomial bounds on the $\gamma_p$-surface area of the convex body $Q \subset \mathbb{R}^n$, it is enough to consider only the portion of $\partial Q$ which is contained in the concentration annulus.

We can also obtain a rough bound for the $\gamma_p$-surface area of a convex body. Namely,

$$\begin{aligned}\gamma_p(\partial Q) &= \frac{1}{n\nu_n J_{n-1,p}} \int_{\partial Q} e^{-\frac{|x|^p}{p}} dx = \frac{1}{n\nu_n J_{n-1,p}} \int_0^\infty t^{p-1} e^{-\frac{t^p}{p}} |\partial Q \cap tB_2^n| dt \\ &\le \frac{J_{n+p-2,p}}{J_{n-1,p}} \approx n^{1-\frac{1}{p}}, \quad n \to \infty.\end{aligned}$$

This bound is far from the best possible. The next section is devoted to the best possible asymptotic upper bound.

## 3. The upper bound

We will use the approach developed by Nazarov in [9]. Let us consider a "polar" coordinate system $x = X(y, t)$ in $\mathbb{R}^n$ with $y \in \partial Q$, $t > 0$. Then

$$\int_{\mathbb{R}^n} \varphi_p(y) d\sigma(y) = \int_0^\infty \int_{\partial Q} D(y, t)\varphi_p(X(y, t)) d\sigma(y) dt,$$

where $D(y, t)$ is the Jacobian of $x \to X(y, t)$. Define

$$\xi(y) = \varphi_p^{-1}(y) \int_0^\infty D(y, t)\varphi_p(X(y, t)) dt. \tag{10}$$

Then

$$1 = \int_{\partial Q} \varphi_p(y)\xi(y) dy,$$

and thus

$$\int_{\partial Q} \varphi_p(y) dy \le \frac{1}{\min\limits_{y\in\partial Q} \xi(y)}. \tag{11}$$

Following [9], we shall consider two such systems. We will be using subindices for $X$, $D$ and $\xi$ to distinguish between the two corresponding systems.

### 3.1. The first coordinate system

Consider the "radial" polar coordinate system $X_1(y, t) = yt$. The Jacobian is $D_1(y, t) = t^{n-1}|y|\alpha$, where $\alpha = \alpha(y)$ denotes the absolute value of the cosine of the angle between $y$ and $\nu_y$ (the unit outer normal vector at $y$). From (10),

$$\begin{aligned} \xi_1(y) &= e^{\frac{|y|^p}{p}} \alpha |y|^{1-n} J_{n-1,p} \\ &\approx \sqrt{\frac{2\pi}{p}} e^{\frac{|y|^p}{p}} \alpha |y|^{1-n} n^{\frac{1}{p}-\frac{1}{2}} e^{F\left((n-1)^{\frac{1}{p}}\right)}, \quad \text{as } n \to \infty, \end{aligned} \tag{12}$$

where $F(t) = (n-1)\log t - \frac{t^p}{p}$. Since $(n-1)^{\frac{1}{p}}$ is the maximum point for $F(t)$, we get that $F\left((n-1)^{\frac{1}{p}}\right) \ge F(|y|)$ for all $y \in \mathbb{R}^n$. So we can estimate (12) from below by

$$\xi_1(y) \gtrsim \sqrt{\frac{2\pi}{p}} n^{\frac{1}{p}-\frac{1}{2}} \alpha. \tag{13}$$

### 3.2. The second coordinate system

Now consider the "normal" polar coordinate system $X_2(y, t) = y + t\nu_y$. Then $D_2(y, t) \ge 1$ for all $y \notin Q$. Thus, by the law of cosines,

$$|y + t\nu_y| \le \sqrt{|y|^2 + t^2 + 2t|y|\alpha}.$$

Thus,

$$\xi_2(y) \ge e^{\frac{|y|^p}{p}} \int_0^\infty e^{-\frac{(|y|^2+t^2+2t|y|\alpha)^{\frac{p}{2}}}{p}} dt. \tag{14}$$

Note that for any positive function $f(x)$ defined on the interval $I$, and any $t_0 \in I$,

$$\int_I e^{-f(t)} dt \ge e^{-f(t_0)} |\{t : f(t) < f(t_0)\} \cap I|. \tag{15}$$

Consider

$$f(t) = \frac{(|y|^2 + t^2 + 2t|y|\alpha)^{\frac{p}{2}}}{p}.$$

By the intermediate value theorem there is $t_1$ such that

$$(|y|^2+t_1^2+2t_1|y|\alpha)^{\frac{p}{2}}=|y|^p+1. \tag{16}$$

Since $f(t)$ is increasing, from (15) and (16) we get

$$\xi_2(y)\gtrsim e^{-\frac{1}{p}}t_1.$$

Now we need to estimate $t_1$ from below. By Remark 1, it suffices to consider $y\in A_p$. For such $y$, note that the Mean Value Theorem yields

$$-|y|^2+(|y|^p+1)^{\frac{2}{p}}\approx\frac{2}{p}|y|^{2-p}. \tag{17}$$

Using (16) and (17), we get

$$t_1=\sqrt{\alpha^2|y|^2-|y|^2+(|y|^p+1)^{\frac{2}{p}}}-\alpha|y|\approx\sqrt{\alpha^2|y|^2+\frac{2}{p}|y|^{2-p}}-\alpha|y|.$$

Multiplying the last expression by its conjugate and applying the inequality $\sqrt{a+b}\le\sqrt{a}+\sqrt{b}$, we get

$$\xi_2(y)\ge e^{-\frac{1}{p}}\sqrt{\frac{2}{p}}|y|^{1-\frac{p}{2}}\frac{1}{1+\sqrt{2p}\alpha|y|^{\frac{p}{2}}}. \tag{18}$$

Now we shall combine estimates (13) and (18) with the assumption that $|y|\in A_p$:

$$\xi(y):=\xi_1(y)+\xi_2(y)\gtrsim n^{\frac{1}{p}-\frac{1}{2}}\left(\sqrt{\frac{2\pi}{p}}\alpha+\frac{C_1(p)}{\alpha\sqrt{n}+1}\right). \tag{19}$$

Note that (19) is minimized whenever $\alpha=C'(p)n^{-\frac{1}{4}}$. The minimal value of (19) is $C(p)^{-1}n^{\frac{1}{p}-\frac{3}{4}}$. Together with (11), this implies that

$$\gamma_p(\partial Q\cap A_p)\le C(p)n^{\frac{3}{4}-\frac{1}{p}}.$$

One can note that $C(p)$ tends to infinity while $p$ tends to infinity or to zero. Finally, we use the above together with Remark 1 to finish the proof of the upper bound part of Theorem 1.

## 4. The lower bound

Let us consider $N$ uniformly distributed independent random vectors $x_i\in S^{n-1}$. Let $\rho=n^{\frac{1}{p}-\frac{1}{4}}$ and $r=r_w=n^{\frac{1}{p}}+w$, where $w\in[-W,W]$, and $W=n^{\frac{1}{p}-\frac{1}{2}}$. Consider a random polytope $Q$ in $\mathbb{R}^n$, defined as follows:

$$Q=\{x\in\mathbb{R}^n:\ \langle x,x_i\rangle\le\rho,\ \forall i=1,\dots,N\}.$$

Let $q(t)$ be the probability that the fixed point on the sphere of radius $\sqrt{t^2+\rho^2}$ is separated from the origin by the hyperplane $\langle x,x_i\rangle=\rho$. Consider a point $x$ belonging to the intersection of $\sqrt{t^2+\rho^2}\cdot S^{n-1}$ and the hyperplane $\langle x_1,x\rangle=\rho$. Since the vectors $x_2,\dots,x_N$ are independent, the probability that $x$ is not separated from the polytope by any of the remaining $N-1$ hyperplanes is $(1-q(t))^{N-1}$. So the expected value of $\gamma_p(\partial Q)$ is

$$\frac{1}{n\nu_n J_{n-1,p}}N\int_{\mathbb{R}^{n-1}}\exp\left(-\frac{(|y|^2+\rho^2)^{\frac{p}{2}}}{p}\right)(1-q(|y|))^{N-1}dy, \tag{20}$$

Passing to polar coordinates, making a change of variables and truncating the integral, we shall estimate (20) asymptotically from below by

$$\frac{\nu_{n-1}}{\nu_n J_{n-1,p}}N\int_{-W}^{W}\left(n^{\frac{1}{p}}+w\right)^{n-2}e^{-\frac{\left(\left(n^{\frac{1}{p}}+w\right)^2+\rho^2\right)^{\frac{p}{2}}}{p}}(1-q(r_w))^{N-1}dw. \tag{21}$$

Note that $\frac{\nu_{n-1}}{\nu_n}\approx\frac{\sqrt{n}}{\sqrt{2\pi}}$. Also note that $\left(n^{\frac{1}{p}}+w\right)^{n-2}\le C_p n^{\frac{n-2}{p}}e^{wn^{1-\frac{1}{p}}}$ for a positive constant $C_p$ depending on $p$ only $\left(\text{since } |w|\le W=n^{\frac{1}{p}-\frac{1}{2}}\right)$. Using the above facts together with Lemma 2, we estimate (21) from below by

$$C'_p n^{1-\frac{2}{p}}e^{\frac{n}{p}}N\int_{-W}^{W}e^{-\frac{\left(\left(n^{\frac{1}{p}}+w\right)^2+\rho^2\right)^{\frac{p}{2}}}{p}}e^{wn^{1-\frac{1}{p}}}(1-q(r_w))^{N-1}dw. \tag{22}$$

**Claim.**

$$e^{-\frac{\left(\left(n^{\frac{1}{p}}+w\right)^2+\rho^2\right)^{\frac{p}{2}}}{p}} \gtrsim C(p)e^{-\frac{n}{p}}e^{-\frac{\sqrt{n}}{2}}e^{-wn^{1-\frac{1}{p}}}.$$

**Proof.** Plugging in the value $n^{\frac{1}{p}-\frac{1}{4}}$ for parameter $\rho$, and estimating $w$ with $W = n^{\frac{1}{2}-\frac{1}{p}}$, as announced earlier, we obtain that

$$e^{-\frac{\left(\left(n^{\frac{1}{p}}+w\right)^2+\rho^2\right)^{\frac{p}{2}}}{p}} \gtrsim e^{-\frac{n}{p}}e^{\frac{n}{p}\left(1-\left(1+2wn^{-\frac{1}{p}}+n^{-\frac{1}{2}}+n^{-1}\right)^{\frac{p}{2}}\right)}.$$

The claim now follows from the fact (obtained by the Mean Value Theorem) that

$$\left(1+2wn^{-\frac{1}{p}}+n^{-\frac{1}{2}}+n^{-1}\right)^{\frac{p}{2}}-1 \le \frac{p}{2}n^{-\frac{1}{2}}+pwn^{-\frac{1}{p}}+O\left(\frac{1}{n}\right). \quad \square$$

Using the above claim we get that (22) is greater than

$$C'(p)n^{1-\frac{2}{p}}e^{-\frac{\sqrt{n}}{2}}N\int_{-W}^{W}(1-q(r_w))^{N-1}dw. \tag{23}$$

Next, we estimate the probability $q(r)$ the same way as in [9]: the probability that a point on a sphere is separated from the polytope by a hyperplane is the quotient of the area of the cap to the area of the whole sphere. We recover both of these by integrating over the "circles" (the spheres with one fewer dimension) and using the Fubini Theorem. Thus,

$$q(r) = \left(\int_{-\sqrt{r^2+\rho^2}}^{\sqrt{r^2+\rho^2}}\left(1-\frac{t^2}{r^2+\rho^2}\right)^{\frac{n-3}{2}}dt\right)^{-1} \times \int_{\rho}^{\sqrt{r^2+\rho^2}}\left(1-\frac{t^2}{r^2+\rho^2}\right)^{\frac{n-3}{2}}dt. \tag{24}$$

By the Laplace method, the first integral is approximately equal to $\sqrt{2\pi}n^{\frac{1}{p}-\frac{1}{2}}$.

Using the elementary inequality $1-a \le e^{-\frac{a^2}{2}}e^{-a}$ for all $a > 0$ (which is true since the function $f(a) = e^{-\frac{a^2}{2}-a}$ is convex for all positive $a$, its derivative at zero is $-1$, and its value at zero is 1), one can estimate the second integral in (24) by

$$\int_{\rho}^{\infty}\exp\left(-\frac{n-3}{4(r^2+\rho^2)^2}t^4\right)\cdot\exp\left(-\frac{n-3}{r^2+\rho^2}\frac{t^2}{2}\right)dt \le \exp\left(-\frac{n-3}{4(r^2+\rho^2)^2}\rho^4\right)\int_{\rho}^{\infty}\exp\left(-\frac{n-3}{r^2+\rho^2}\frac{t^2}{2}\right)dt.$$

The first multiple is of order $e^{-\frac{1}{4}}$ under our assumptions on $r$ and $\rho$. The second integral can be estimated with usage of the inequality

$$\int_{\rho}^{\infty}e^{-a\frac{t^2}{2}}dt \le \frac{1}{a\rho}e^{-a\frac{\rho^2}{2}}, \quad a > 0,\ \rho > 0.$$

We note that under our assumptions on $\rho$ and $W$, $a\rho^2 = \frac{n-3}{\rho^2+r^2}\rho^2$ is of order $n^{\frac{1}{2}}\left(1-3n^{-\frac{1}{2}}\right)$ up to an additive error $\sim n^{-\frac{1}{2}}$. Hence, one can estimate $q(r)$:

$$q(r) \le Cn^{-\frac{1}{4}}e^{-\frac{\sqrt{n}}{2}}. \tag{25}$$

Now, one can choose $N = Cn^{\frac{1}{4}}e^{\frac{\sqrt{n}}{2}}$. From (23) and (25) it follows that the expectation of $\gamma_p(\partial Q)$ is greater than

$$C''(p)n^{1-\frac{2}{p}}e^{-\frac{\sqrt{n}}{2}}n^{\frac{1}{4}}e^{\frac{\sqrt{n}}{2}}\cdot 2W. \tag{26}$$

Plugging in $W = n^{\frac{1}{p}-\frac{1}{2}}$, we get $c(p)n^{\frac{3}{4}-\frac{1}{p}}$ for the lower bound for the expectation of $\gamma_p(\partial Q)$, which finishes the proof of Theorem 1. $\square$

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