Each problem is worse 25 points. To get the full score (100 points) please select 4 out of 6 problems of your choice and solve them. Please indicate which problems you select at the first page of the exam; only those problems shall be graded. This scheme is used for qualifying exams, and the problems at the qualifying exam shall be similar in difficulty.

1. Let $p \geq 1$. Show that if $E|X_n - X|^p \to 0$, and $X_n$ converges to $Y$ almost surely, then $X = Y$ almost surely.

2. Consider the sample space $\Omega = \{0, 1, 2\} \times \{0, 1, 2\}$ with uniform probability measure on $\Omega$. Using the notation $\omega = (\omega_1, \omega_2)$, consider random variables $X(\omega) = \omega_1$ and $Y(\omega) = \omega_2$. Define $A = X$, $B = (X + Y) \mod 3$ and $C = (X + 2Y) \mod 3$. Show that $A, B, C$ are pairwise independent, but not jointly independent.

3. For a $p > 0$, let $X = (X_1, ..., X_n)$ be a random vector distributed according to the density $f(x) = 1_Q \cdot (p + 1)^n \cdot \prod_{i=1}^{n} x_i^p$, where $Q = [0, 1]^n = \{x \in \mathbb{R}^n : x_i \in [0, 1] \forall i = 1, ..., n\}$. Assume that for every $\delta > 0$ there exists a positive integer $N$ so that for all $n \geq N$, $P(|X| \in \sqrt{n}[\sqrt{\frac{3}{2}} - \delta, \sqrt{\frac{3}{2}} + \delta]) \geq 0.1$. Find $p$.

4. Assume that $P(\lim sup_{n \to \infty} A_n) = 1$ and $P(\lim inf_{n \to \infty} B_n) = 1$. Prove that $P(\lim_{n \to \infty} (A_n \cap B_n)) = 1$.

5. Suppose $X_n, n = \{1, 2, ...\}$ and $X$ are random variables with bounded first moments. Assume that $X_n \geq 0$ almost surely, $EX_n = 1$ and $E(X_n \log X_n) \leq 1$. Assume that for every bounded random variable $Y$, $E(X_n Y) \to_{n \to \infty} E(XY)$. Show that:
   - $X \geq 0$ almost surely;
   - $EX = 1$;
   - $E(X \log X) \leq 1$.

6. Let $X_1, X_2, ...$ be independent random variables with $EX_i = \mu_i < \infty$ and $\text{Var}(X_i) = \sigma_i^2 < \infty$. Let $S_n = X_1 + ... + X_n$. Show that
   $$P\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \mu_i \right| \geq t \right) \leq \frac{1}{t^2} \sum_{i=1}^{n} \sigma_i^2.$$