HOME WORK 4. PROBABILITY I, FALL 2016.

1. Let $X_1, ..., X_n, ...$ be independent Poisson random variables with $\mathbb{E}X_n = \lambda_n$. Let $S_n = X_1 + ... + X_n$. Show that if $\sum_{n=1}^{\infty} \lambda_n = \infty$, then $\frac{S_n}{\mathbb{E}S_n} \to 1$ almost surely.

2. Let A_n be a sequence of independent events with $P(A_n) < 1$ for all n. Show that $P(\cup A_n) = 1$ implies $P(A_n \text{ i.o.}) = 1$.

3. Given a sequence of numbers $p_n \in [0, 1]$, let $X_1, ..., X_n, ...$ be independent random variables with $P(X_n = 1) = p_n$ and $P(X_n = 0) = 1 - p_n$. Show that
a) $X_n \to 0$ in probability if and only if $p_n \to 0$;
b) $X_n \to 0$ almost surely if and only if $\sum p_n < \infty$.

4. Let X_0 be a random vector in $\mathbb{R}^2$ taking the value $(1, 0)$ with probability 1. Define inductively X_{n+1} as a random vector uniformly distributed in the disc of radius $|X_n|$ centered at the origin. Prove that $\frac{\log |X_n|}{n} \to c$ almost surely, and find the value of c.

5. Prove the Stirling's formula, that is,

$$n! = (1 + o(1)) \sqrt{2\pi n} n^n e^{-n},$$

as $n \to \infty$.

6. Let $X_1, ...$ be a sequence of i.i.d. Poisson random variables with $\lambda = 1$, and let $S_n = X_1 + ... + X_n$. Show that

$$\sqrt{2\pi n} \cdot P(S_n = k) \to e^{-\frac{x^2}{2}},$$

where $\frac{k-n}{\sqrt{n}} \to x$.

7. Show that if $F_n \to^w F$, and F is continuous, then $\sup_x |F_n(x) - F(x)| \to 0$, as $n \to \infty$.

8. Show that if $\varphi(t)$ is a characteristic function, then $\Re\varphi(t)$ and $|\varphi(t)|^2$ are also characteristic functions.

9. Show that if the characteristic function of a random variable X takes only real values, then X and $-X$ are identically distributed.

10. Let random variable X have a density $f(x) = \frac{1}{\pi(1+x^2)}$ on $\mathbb{R}$.
a) Find the characteristic function of X.
b) Let $X_1, X_2, ...$ be independent copies of X and let $S_n = X_1 + ... + X_n$. Show that $\frac{S_n}{n}$ has the same distribution as X_1.