1. Let \( X_1, \ldots, X_n, \ldots \) be independent Poisson random variables with \( \mathbb{E} X_n = \lambda_n \). Let \( S_n = X_1 + \ldots + X_n \). Show that if \( \sum_{n=1}^\infty \lambda_n = \infty \), then \( \frac{S_n}{\mathbb{E} S_n} \to 1 \) almost surely.

2. Let \( A_n \) be a sequence of independent events with \( P(A_n) < 1 \) for all \( n \). Show that \( P(\bigcup A_n) = 1 \) implies \( P(A_n \text{ i.o.}) = 1 \).

3. Given a sequence of numbers \( p_n \in [0, 1] \), let \( X_1, \ldots, X_n, \ldots \) be independent random variables with \( P(X_n = 1) = p_n \) and \( P(X_n = 0) = 1 - p_n \). Show that
   a) \( X_n \to 0 \) in probability if and only if \( p_n \to 0 \);
   b) \( X_n \to 0 \) almost surely if and only if \( \sum p_n < \infty \).

4. Let \( X_0 \) be a random vector in \( \mathbb{R}^2 \) taking the value \( (1, 0) \) with probability 1. Define inductively \( X_{n+1} \) as a random vector uniformly distributed in the disc of radius \( |X_n| \) centered at the origin. Prove that \( \frac{\log |X_n|}{n} \to c \) almost surely, and find the value of \( c \).

5. Prove the Stirling’s formula, that is,
   \[ n! = (1 + o(1)) \sqrt{2\pi n} n^ne^{-n}, \]
as \( n \to \infty \).

6. Let \( X_1, \ldots \) be a sequence of i.i.d. Poisson random variables with \( \lambda = 1 \), and let \( S_n = X_1 + \ldots + X_n \). Show that
   \[ \sqrt{2\pi n} \cdot P(S_n = k) \to e^{-\frac{k^2}{2}}, \]
where \( \frac{k-n}{\sqrt{n}} \to x \).

7. Show that if \( F_n \to^w F \), and \( F \) is continuous, then \( \sup_x |F_n(x) - F(x)| \to 0 \), as \( n \to \infty \).

8. Show that if \( \varphi(t) \) is a characteristic function, then \( \Re \varphi(t) \) and \( |\varphi(t)|^2 \) are also characteristic functions.
9. Show that if the characteristic function of a random variable $X$ takes only real values, then $X$ and $-X$ are identically distributed.

10. Let random variable $X$ have a density $f(x) = \frac{1}{\pi(1+x^2)}$ on $\mathbb{R}$.
   a) Find the characteristic function of $X$.
   b) Let $X_1, X_2, ...$ be independent copies of $X$ and let $S_n = X_1 + ... + X_n$. Show that $\frac{S_n}{n}$ has the same distribution as $X_1$. 