HOME WORK 5, PROBABILITY I

1. Show that if $X_1, ..., X_n$ are i.i.d. with Cauchy distribution then $\frac{X_1+...+X_n}{n}$ has the same Cauchy distribution too.

2. Show that if $\mathbb{E}|X|^n < \infty$ then the characteristic function φ of X has a continuous derivative of order n given by

$$\varphi^{(n)}(t) = \mathbb{E}(i^n X^n e^{itX}).$$

3. Show that if Y_n are random variables with characteristic functions $\varphi_n(t)$, then $Y_n \to^w 0$ if and only if there is a $\delta > 0$ so that $\varphi_n(t) \to 1$ for all $|t| < \delta$.

4. Suppose that $X_1, ...$ are i.i.d. with $\mathbb{E}X_i = 0$. Suppose $\frac{X_1+...+X_n}{\sqrt{n}}$ converges weakly to a limit. Prove that $\mathbb{E}X_i^2 < \infty$.

5. Let $X_1, ..., X_n, ...$ be i.i.d. with $\mathbb{E}X_i = 0$ and $\mathbb{E}X_i^2 = \sigma^2 \in (0, \infty)$. Show that

$$\frac{\sum_{i=1}^n X_i}{\sqrt{\sum_{i=1}^n X_i^2}} \to^w Z,$$

where Z is a standard normal random variable.

6. Show that if a characteristic function $\varphi(t) = 1 + o(t^2)$ as $t \to 0$ then $\varphi(t) = 1$ everywhere.

7. Suppose $P(X_m = -m) = P(X_m = m) = \frac{m^{-2}}{2}$, and for $m \geq 2$,

$$P(X_m = 1) = P(X_m = -1) = \frac{1 - m^{-2}}{2}.$$

Show that $\frac{Var(S_n)}{n} \to 2$ but $\frac{S_n}{\sqrt{n}} \to^w Z$ Where Z is gaussian. Which condition of the CLT for triangular arrays is not satisfied?

8. Let T be a random variable taking values on $[0, \infty)$, such that $P(T > s + t, T > s) = P(T > t)P(T > s)$, and $P(T > t)$ is differentiable in t. Show that there is a $\lambda > 0$ so that $P(T > t) = e^{-\lambda t}$.