Week 1 Worksheet

Sections from Thomas 13th edition: 12.4, 12.5, 12.6, 13.1

Exercises

1. A plane is a set of points that satisfies an equation of the form \(c_1 x + c_2 y + c_3 z = c_4\).
   (a) Find any three distinct points, \(P, Q, R\), in the plane \(2x + 3y + z = 0\).
   (b) Use a cross product to construct a vector perpendicular to the plane.
   (c) How is the vector you created related to the vector made from the coefficients of \(x, y, \) and \(z\) in the equation for the plane, which is \(2\hat{i} + 3\hat{j} + \hat{k}\)?
   (d) Find any point that is not in the plane, \(S\), and then find the closest point on the plane to \(S\).

2. Calculate the cosine of the angle between the vector \(\vec{x} = \hat{i} + 2\hat{k}\) and the plane \(2x + y + z = 0\). How are the plane and the vector related to each other, geometrically? Why?

3. Find the distance between the point \((1,2,3)\) and the line \(\vec{r} = \hat{i} + 2\hat{k} + t(\hat{i} - 2\hat{j} + 3\hat{k})\).

4. Consider the line \((-1, t, 1)\) and the plane \(2x + y - z = 3\), where \(t \in \mathbb{R}\). Find all values of \(t\) so that the distance between points on the line and the plane is \(2\sqrt{6}\).

5. Consider the points \(P(1,0,3), Q(2,2,3), R(0,0,-1)\).
   (a) Find a vector that is perpendicular to the plane that passes through these points.
   (b) Find a point \(S\) so that a unique plane that passes through \(P, Q, \) and \(S\) cannot be found. Describe why you cannot find a unique plane that passes through \(P, Q, \) and your point, \(S\).

6. Find the equation of the plane that passes through the point \((-1,2,1)\) and contains the line \(x = y = z\).

7. Suppose we have two planes \(2x + y - z = 3\) and \(x + 3y + z = 0\). Find the line of intersection between these two planes, and find the equation of the plane that passes through the line of intersection and through the point \((0,0,0)\).

Group Work Problems

1. Consider the set of all points that are equidistant from the point \((0,1,0)\) and the plane \(z = 1\).
   (a) Find an equation that represents the surface.
   (b) Briefly describe the surface in words.
   (c) Give a rough sketch of the surface.

2. Consider the surfaces \(z = \sqrt{4 - x^2 - y^2}\) and \(y^2 + x^2 - 2y = 0\).
   (a) Find a parametric vector representation, \(\vec{r}(t)\), of the curve that satisfies both equations.
   (b) Give a rough sketch of the surfaces and the curve in one plot.
Additional Problems (if time permits)

1. Find the area of the triangle with the vertices (6,3), (4,5), and (3,4).

2. Indicate which of the following are always true.
   (a) \( \vec{a} \cdot \vec{a} = |\vec{a}| \)
   (b) \( \vec{a} \times (-\vec{a}) = \vec{0} \)
   (c) \( (\vec{a} \times \vec{b}) \cdot \vec{a} = 0 \)

3. Which of the following make sense? Explain why/why not.
   (a) \( \vec{a} \times (\vec{b} \cdot \vec{c}) \)
   (b) \( \vec{a} \cdot (\vec{b} \cdot \vec{c}) \)
   (c) \( \vec{a} \times (\vec{b} \times \vec{c}) \)
   (d) \( \vec{a} \cdot (\vec{b} \times \vec{c}) \)

4. Consider the surface \( z = ax^2 + by^2 \), where \( a \) and \( b \) are constants. Identify all possible surfaces for the following cases, by stating the name of the surface and describing its orientation, if applicable.
   (a) \( ab > 0 \)
   (b) \( ab < 0 \)
   (c) \( a = b = 0 \)

5. The path of an object is given by \( \vec{r} = 2t\hat{i} + t^2\hat{j} \) for \( t \geq 0 \). Sketch the curve in the \( xy \)-plane and indicate the direction of motion.

6. \( S \) is the surface described by the equation \( x^2 - 2x - y - z^2 - 4z = 3 \). Create a rough sketch of \( S \) and name the surface.
Worksheet 2

Sections from Thomas 13th edition: 13.2, 13.3, 13.4

Exercises

1. An object is moving along the curve \( \mathbf{r}(t) = 5 \sin(\pi t) \hat{i} + 5 \cos(\pi t) \hat{j} + 12\pi t \hat{k}, t \in \mathbb{R} \). Where could the object be, if it has travelled a distance of \( 39\pi \) units along the curve, starting from \( P(0, -5, -12\pi) \)?

2. An object is moving in the \( yz \)-plane. At point \( P \), its tangent vector is \( \mathbf{T} = \hat{k} \). What could the normal vector be equal to at \( P \)? List all possibilities. \( \text{Note: a similar question was on a Spring 2016 final exam.} \)

3. An object is moving in the \( xy \)-plane along the curve shown in the graph. The direction of motion is indicated with arrows. Identify the points where the curvature has a local maximum. Sketch the normal and tangent vectors at those points.

![Graph of curve](image)

4. An object is moving along the curve \( \mathbf{r}(t) = -9 \ln(\sec t) \hat{i} - 9 \hat{j}, 0 \leq t \leq \pi/4 \).
   
   (a) Compute the arc length parametrization.
   (b) Compute the unit tangent vector.
   (c) Compute the curvature.

5. A ball is thrown from a height of 3 meters above the ground, at a speed of \( 2\sqrt{2} \) m/s, at an angle of \( 45^\circ \) above the horizontal. A constant wind is blowing that adds a component of \( 4 \hat{i} \) m/s\(^2\) to the ball’s acceleration while it is in the air. Where does the ball land? You may use \( g = 10 \) m/s\(^2\).

Group Work Problems

1. A golfer can send a golf ball 300m across a level ground. From the tee in the figure, can the golfer clear the water? \( \text{You may use } g = 10 \text{ m/s}^2, \text{but either way you may want to use a calculator.} \)

![Golf course diagram](image)

2. An object is moving along the intersection of the plane \( y = 2 \) and the surface \( z = x^2 + y^2 \).
   
   (a) Find a parametric representation for the motion.
(b) Sketch of the plane, surface, and the curve. Without any calculation, indicate the point(s) where you think the curvature, $\kappa$, of your curve is maximized, and sketch the normal and tangent vectors at that point.

(c) Verify your answer to the previous question by finding the point of maximum $\kappa$, and calculate the tangent vector at that point. Hint: some curvature formulas require fewer calculations than others.

(d) Find the equation of the osculating circle at the point where $\kappa$ is maximum.

**Additional Problems (if time permits)**

1. Without any calculation, what is the value of the normal vector in the second group work problem? Calculate the normal vector for this problem when $t = 0$ to verify your hypothesis.
Worksheet 3
Sections from Thomas 13th edition: 14.1, 14.2, 14.3

Exercises

1. For the following function, a) identify and sketch the domain, b) state the boundary of the domain, c) indicate whether the domain is open and/or closed, and d) determine the range of \( g(x, y) \).

\[
g(x, y) = \frac{\sqrt{y + 1}}{x^2 y + xy^2}
\]

2. (a) Give an example of a function whose domain is neither open nor closed.
   (b) Give an example of a function of two variables, \( f(x, y) \), whose level curves, \( C = f(x, y) \), are a family of parabolas that are symmetric about the \( y \)-axis.
   (c) Give an example of a function of two variables, \( g(x, y) \), whose domain is an open set, and whose level curves, \( C = g(x, y) \), are straight lines with slope \( C \).
   (d) Give an example of a function of three variables, \( h(x, y, z) \), whose level surfaces are a family of cones whose vertices are located at the origin.

3. An object is moving along the intersection of the plane \( y = 3 \) and the surface \( f(x, y) = x^2 + y^2 \). Sketch the plane, surface, and path. Find the equation of the tangent line to the path at the point \( P(1, 3, 10) \), express the line in parametric vector form, and add the tangent line to your sketch.

4. Question 66 from Section 14.3, which is: find the value of \( \frac{\partial z}{\partial y} \) at \((1, -1, -3)\) if \( xz + y \ln x - x^2 + 4 = 0 \) defines \( x \) as a function two independent variables \( y \) and \( z \).

5. Show that the limit does not exist.

\[
\lim_{(x, y) \to (1, 0)} \frac{x(x - 1)^3 + y^2}{4(x - 1)^2 + 9y^3}
\]

Group Work Problems

1. Identify where \( f(x, y) \) is continuous, where

\[
f(x, y) = \begin{cases} 
\frac{x^2 - (y - 1)^2}{x^2 + (y - 1)^2} & \text{if } (x, y) \neq (0, 1) \\
0 & \text{if } (x, y) = (0, 1)
\end{cases}
\]

2. Shown below is a function, \( f(x, y) \), and \( f_y(x, y) \), on the domain \(-4 \leq x \leq 4, -4 \leq y \leq 4\).

   i) Which of the two surfaces is \( f_y(x, y) \)?
   ii) Explain your reasoning.
Worksheet 4

Sections from Thomas 13th edition: 14.4, 14.5, 14.6

Exercises

1. Below is a contour map for a function $f(x, y)$. At points P and Q, draw the normalized gradient vector

$$\vec{g}(x, y) = \frac{\nabla f}{|\nabla f|}$$

Note: this problem is based on a question from a 2015 Math 2401 quiz created by Dr. Tom Morley.

2. Consider $f(x, y, z) = xz + y^2$.
   
   (a) Determine the maximum rate of change of $f(x, y, z)$, and the direction in which it occurs, at the point $(1, 1, 2)$.
   
   (b) If $x = \cos t$, $y = t + 1$, $z = t + 2$, use the chain rule to find $\frac{df}{dt}(x(t), y(t), z(t))$ at the point $(1, 1, 2)$.
   
   (c) Approximate the value of $f(1, 0.9, 2.1)$ with a linear approximation.

3. Find the directional derivative of $f = z \ln(x/y)$ at $P(1, 1, 2)$ towards the point $Q(2, 2, 1)$. Provide a geometric interpretation of your derivative.

4. The radius of a cylinder is decreasing at a rate of 2 cm/s while its height is increasing at a rate of 3 cm/s. At what rate is the volume changing when the radius is 10 cm and the height is 100 cm?

5. Calculate $\frac{du}{dt}$ given that $u = x^2 - y^2$, $x = t^2 - 1$, and $y = 3 \sin(\pi t)$.

6. Consider the surface $x^2yz + xy - y^2z^2 = -27$.
   
   (a) Construct an equation of the tangent plane to the surface at $P(1, 3, 2)$.
   
   (b) Construct a parametrization of the normal line at $P(1, 3, 2)$.

7. Consider the surface $z = x^3y - x^2y^2$. Construct two different vectors that are normal to this surface at $P(2, 1, 4)$.

8. Let $f(x, y, z) = e^x + \cos(y + z)$. Compute the linearization of $f$ at $P(0, \pi/4, \pi/4)$. 
Group Work Problems

1. A closed rectangular box 2 inches long, 0.5 inches wide, 4 inches high, is covered by a coat of paint \( \frac{1}{16} \) inches thick. Use a differential to estimate the amount of paint on the box.

2. Let \( z = f(x, y) = \ln(4x^2 + y^2) \). Note: this question is based on a 2016 Math 2550 midterm question.
   
   (a) Identify all possible unit direction vectors, \( \vec{u} \), that satisfy \( D \vec{u} f(1, 1) = 0 \).

   (b) Identify all points, if any, on \( f(x, y) = \ln(4x^2 + y^2) \), where the tangent plane is parallel to the plane \( 4x - z = 3 \).
Worksheet 5

Sections from Thomas 13th edition: 14.7, 14.8, 14.9

Exercises

1. Find the dimensions of a rectangular box of maximum volume such that the sum of its 12 lengths is a constant $L$.

2. Calculate the extreme values of the function $f(x, y) = x^2 + 4y^2 + x - 2y$ on the closed region bounded by $x^2 + 4y^2 = 4$.

3. Consider the function $f(x, y) = 3xy - x^3 - y^3$. Find the points where the tangent plane is horizontal, find the critical points of $f(x, y)$, and classify the critical points as min, max, or saddle points.

4. 14.9.4: Use Taylor’s formula for $f(x, y)$ at the origin to find quadratic and cubic approximations to $f$ near the origin.

\[ f(x, y) = \sin(x)\cos(y) \]

Group Work Exercises

1. Use Lagrange Multipliers to calculate the dimensions of an aluminum can in the shape of a cylinder, whose volume is $V$, and whose surface area is minimum.

2. Problem 41 in section 14.7: A flat circular plate lies in the region $x^2 + y^2 \leq 1$. The plate, including the boundary, is heated so that the temperature at the point $(x, y)$ is $T(x, y) = x^2 + 2y^2 - x$. Find the temperatures at the hottest and coldest points on the plate.
Worksheet 6

Sections from Thomas 13th edition: 15.1, 15.2, 15.3, 15.4

Exercises

1. 15.4 Problem 32: Calculate the area common to the interiors of the cardioids \( r = 1 + \cos \theta \) and \( r = 1 - \cos \theta \).

2. (a) Construct a double integral that represents the volume of the solid enclosed by the cylinder \( x^2 + y^2 = 1 \), the planes \( z = 1 - y \), \( x = 0 \), \( z = 0 \), in the first octant. Use the integration order \( dy \, dx \).
   
   (b) Change the order of integration to the problem in part (a).

3. Calculate the area of region \( R \) in the \( xy \)-plane for which the integral is maximum.

   \[
   \iint_{R} \left( 9 - \frac{x^2}{4} - 4y^2 \right) \, dA
   \]

4. Evaluate the double integral.

   \[
   \int_{0}^{4} \int_{y}^{4} e^{x^2} \, dx \, dy
   \]

5. Evaluate the iterated integral by converting to polar coordinates.

   \[
   \int_{0}^{1} \int_{y}^{\sqrt{2-y^2}} x \, dx \, dy
   \]

6. Consider the following iterated integral:

   \[
   \int_{0}^{\sqrt{2}} \int_{x}^{\sqrt{1-x^2}} x \, dy \, dx
   \]

   (a) Sketch the region of integration, \( R \).
   
   (b) Change the order of integration to \( dx \, dy \).
   
   (c) Use polar coordinates to evaluate \( \iint_{R} x \, dA \).

Group Work Problems

1. (2 marks) Convert to a double integral in polar coordinates.

   \[
   \int_{0}^{2} \int_{0}^{\sqrt{4-(x-2)^2}} xy \, dy \, dx
   \]

2. (2 marks) 15.3 Problem 12: Sketch the region bounded by the lines \( y = x - 2 \), \( y = -x \) and the curve \( y = \sqrt{x} \). Construct one or two double integrals that represent the area of the region.

3. (2 bonus marks) Compute the area of the region in the previous problem using one or two double integrals. These bonus marks are graded for accuracy.
I Worksheet 7

Sections from Thomas 13th edition: 15.5, 15.6, 15.7, 15.8

Exercises

1. 15.5 Problem 30: Find the volume of the region in the first octant bounded by the coordinate planes and the surface \( z = 4 - x^2 - y \).

2. A solid is bounded below by \( z = \sqrt{x^2 + y^2} \) and above by \( z = 1 \). The density of the solid at each point \( P \) in the solid is proportional the distance from \( P \) to the \( z \)-axis.
   (a) Calculate the mass of the solid.
   (b) Calculate the center of mass of the solid.

3. Use a change of variables to evaluate \( \iint_{\Omega} (x + y) \, dA \), where \( \Omega \) is the region bounded by
\[
x - y = 0, \quad x - y = \pi, \quad x + 2y = 0, \quad x + 2y = \pi/2
\]

4. Construct a triple integral that represents the volume of the solid bounded above by \( z = 1 \) and below by \( z = \sqrt{x^2 + y^2} \).

5. Construct an integral that represents the volume of the region bounded by \( x^2 + y = 1 \) and \( z^2 + y = 1 \), in the first octant.

6. Set up the triple integral in at least two other ways by changing the integration order.
\[
\int_0^1 \int_0^{\pi/2} \int_0^{1-y} f(x, y, z) \, dz \, dy \, dx
\]

Group Work Problems

1. (2 marks) Calculate the volume of the ice cream cone, bounded below by \( z = \sqrt{3x^2 + 3y^2} \) and bounded above by \( x^2 + y^2 + z^2 = 1 \).

2. (2 marks) A single wedge is cut from a spherical ball of radius \( R \) by two planes that meet in a diameter. The angle between the planes is \( \alpha \in (0, \pi/2) \). Use spherical coordinates to construct a triple integral that represents the volume of the wedge.

3. (bonus 2 marks) In the previous question, what is the volume of the solid that remains? This bonus question is graded for accuracy.