Mathematics 1501 Hour Examination
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Directions: Do all problems. Show your work, and justify your answers. Calculators are allowed, but this is a closed book examination. Please put your name and your recitation leader’s name on each page of your examination.

1. (32) Calculate each of the following derivatives and integrals.

a. \( \frac{d}{dx} (\ln(7x^3)) = \frac{1}{7x^3} \cdot (3 \cdot 7x^2 \cdot 4) = \frac{21}{x} \)

b. \( \int_0^{\pi/4} \tan 2x \, dx = \frac{1}{2} \int_0^{\pi/4} \tan u \, du = \frac{1}{2} \left[ \ln \sec u \right]_0^{\pi/4} + \frac{1}{2} \ln |\sec u| \)

\( = \frac{1}{2} \ln \sqrt{2} = \frac{1}{4} \ln 2 \)

c. \( \frac{d}{dx} (e^{5x^2+1}) = e^{5x^2+1} \cdot (10x) \)

d. \( \int \frac{e^{2x}}{4+e^{2x}} \, dx = \frac{1}{2} \int \frac{du}{4+u} = \frac{1}{2} \ln |4+u| + C \)

\( = \frac{1}{2} \ln (4 + e^{2x}) + C \)

\( u = e^x \)

\( du = 2e^x \, dx \)
2. (20) Let R denote the region in the plane between the graphs of \( y = \sqrt{1 + x^2} \) and \( y = x \) from \( x = 0 \) to \( x = 1 \).

a. Write out an integral, with limits, which represents the volume of the solid generated by revolving R about the y-axis.

\[
\int_0^1 2\pi x \sqrt{1 + x^2} \, dx - 2\pi \int_0^1 x^2 \, dx
\]

b. Evaluate the integral in part a of this problem.

\[
2\pi \int_0^1 x \sqrt{1 + x^2} \, dx - 2\pi \int_0^1 x^2 \, dx
= \left[ 2\pi \frac{1}{3} (1 + x^2)^{3/2} \right]_0^1 - \left[ 2\pi \frac{x^3}{3} \right]_0^1
= \frac{2\pi}{3} (2\sqrt{2} - 1) - \frac{2\pi}{3} = \frac{2\pi}{3} (2\sqrt{2} - 2)
\]

Since \( \int x \sqrt{1 + x^2} \, dx \)

\[
= \int \frac{1}{2} u^{1/2} \, du = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C
\]

\[
= \frac{1}{3} u^{3/2} + C = \frac{1}{3} \left( 1 + x^2 \right)^{3/2}
\]

\[
\begin{align*}
\frac{du}{dx} &= 1 + x^2 \\
\int \frac{2x}{\sqrt{1 + x^2}} \, dx
\end{align*}
\]
3. (24) A water tank is in the shape of a right circular cone with radius 2 and height 6, so that the cone can be pictured as the region generated by revolving the triangle with vertices (0,0), (0,6) and (2,6) about the y-axis. The tank is full of water.

a. How much water is in the tank?

\[
V = 2\pi \int_0^2 x \left(6 - 3x\right) dx = 2\frac{\pi}{3} \left[3x^2 - x^3\right]_0^2 = 2\frac{\pi}{3} (4 - 8) = 8\frac{\pi}{3}
\]

b. How much work is done in pumping all of the water out of the tank? The weight density of water is about 62.5.

Change coordinate system:

\[
A(x) = \pi \left(2 - \frac{x}{3}\right)^2
\]

\[
W = \int_0^6 \left(6.25\right) \pi x \left(2 - \frac{x}{3}\right)^2 dx = \frac{125\pi}{2} \int_0^6 \left[4x - \frac{4}{3} x^2 + \frac{x^3}{9}\right] dx
\]

\[
= \frac{125\pi}{2} \left[2x^2 - \frac{4}{3} \frac{x^3}{3} + \frac{x^4}{54}\right]_0^6 = \frac{125\pi}{2} \left[72 - 96 + 36\right] = \frac{125\pi}{2} (12) = 750 \pi
\]
4. (12) A spring of natural length 9 inches compressed to a length of 6 inches exerts a restoring force of 3 pounds. How much work is done by the spring in restoring itself from 13 inches to its natural length?

\[ F(x) = -kx \]

\[ 3 = -k \left( -\frac{3}{4} \right) = \frac{k}{4}, \quad \text{so} \quad k = 12 \]

\[ W = \int_{3}^{0} (-12x) \, dx = \left[ -6x^2 \right]_{3}^{0} = 6 \left( \frac{1}{3} \right)^2 = \frac{2}{3} \text{ ft-lb} \]

5. (12) Simplify each of the following.

a. \( \ln(2x) - \ln(2) = \ln(2) + \ln(x) - \ln(2) = \ln x \)

b. \( 3^{\log_{3}16} = 4 \quad \text{since} \quad \log_{3}16 = 4 \)

c. \( \log_{3} \sqrt{64} = \log_{2} 8 = 3 \)