Instructions: 1. This is a closed book exam. You may use a calculator. 
2. Show your work and explain your answers and reasoning.

1. (56) Calculate each of the following integrals and derivatives.

a. \( \frac{d}{dx}(e^{x^3}\cos(7x)) \)

\[ = 3e^{x^3}\cos(7x) - 7e^{x^3}\sin(7x) \]

b. (8) \( \int x\sec^2xdx = \int x\,d(tanx) \)

\[ = x\tan x - \int \tan x \, dx \]
\[ = x\tan x - \ln |\sec x| + C \]

c. (8) \( \int_0^3 \frac{4x}{\sqrt{1+x^3}} \, dx = 2\int_0^3 \frac{2x \, dv}{\sqrt{1+x^2}} \)

\[ = 2 \left[ \int_0^3 u^{1/2} \, du \right] \]
\[ = 2 \left[ \frac{2u^{3/2}}{3/2} \right] \]
\[ = \left[ 4\sqrt{u} \right]_0^3 \]
\[ = 2(2 \sqrt{10} - 2 \sqrt{1}) = 4\sqrt{10} - 4 \]
1. (continued) Calculate each of the following integrals.

d. \( \int \cos^3 x \, dx = \int (\cos^2 x \sin x) \, dx \)

\[ = \int (1 - \sin^2 x) \sin x \, dx \]

\[ = \int (1 - u^2) \, du = \int 1 - 2u^2 + u^4 \, du \]

\[ = u - \frac{2}{3} u^3 + \frac{1}{5} u^5 + C \]

\[ = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C \]

e. \( \int \frac{x \, dx}{(x+2)(x-2)} \)

\[ \frac{x}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2} \]

\[ x = A(x+2) + B(x-2) \]

\[ 2 = 4A \quad \Rightarrow \quad A = \frac{1}{2} \]

\[ -2 = -4B \quad \Rightarrow \quad B = \frac{1}{2} \]

\[ \int \frac{x \, dx}{(x-2)(x+2)} = \int \frac{1/2}{x-2} + \frac{1/2}{x+2} \, dx \]

\[ = \frac{1}{2} \ln |x-2| + \frac{1}{2} \ln |x+2| + C \]

\[ = \ln (|x^2 - 4|^{1/2}) + C \]
1. (continued) Calculate each of the following integrals and derivatives.

f. \( (8) \int \frac{2x}{1+x^2} + \frac{5}{1+x^2} - \frac{3}{\sqrt{1-x^2}} \, dx \)

\[ = \ln(1+x^2) + 5 \arctan x - 3 \arcsin x + C \]

\[ \frac{d}{dx} \int_{0}^{x} \cos^3 t \, dt = \cos^2 x \]
2. (16) A plot of land is to be rectangular in shape and to have one side along a very long wall. There are 200 feet of fencing available, and no fencing is needed on the side along the wall. What are the dimensions of the plot with the largest area?

\[
x = 100 - \frac{x}{2}
\]

\[
A(x) = \text{Area} = ax = (100 - \frac{x}{2})x = 100x - \frac{x^2}{2}
\]

\[
A'(x) = 100 - x
\]

\[
A'(x) = 0 \quad \Rightarrow \quad x = 100
\]

The critical values of \( x \) are 0, 100, 200.

Clearly \( x = 0 \) and \( x = 200 \) give a minimum value (namely 0) for the area, so the maximum must occur when \( x = 100 \).

Thus \( x = 100 \) ft and \( a = 50 \) ft.
3. (30) Let \( L \) be the lens shaped region in the first quadrant between the graphs of 
\( y = x^3 \) and \( y = \sqrt[3]{x} \) and between \( x = 0 \) and \( x = 1 \).

a. \((10)\) Find the area of \( L \)

\[
\int_0^1 x^{\frac{1}{3}} - x^3 \, dx = \left[ \frac{3}{4} x^\frac{4}{3} - \frac{x^4}{4} \right]_0^1 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}.
\]

b. \((10)\) If this lens-shaped region \( L \) is revolved around the \( y \)-axis, it sweeps out a solid. What is the volume of this solid?

\[
\int_0^1 2\pi x \left( x^{\frac{4}{3}} - x^3 \right) \, dx = 2\pi \int_0^1 x^{\frac{4}{3}} - x^4 \, dx = 2\pi \left[ \frac{3}{7} x^{\frac{7}{3}} - \frac{x^5}{5} \right]_0^1 = 2\pi \left( \frac{3}{7} - \frac{1}{5} \right)
\]

\[
= 2\pi \left( \frac{15}{35} - \frac{7}{35} \right) = \frac{16\pi}{35}.
\]
3. (continued) Let \( L \) be the lens shaped region in the first quadrant between the graphs of \( y = x^4 \) and \( y = \sqrt{x} \), and between \( x = 0 \) and \( x = 1 \).

c. (10) Find the coordinates of the centroid of this lens-shaped region \( L \).

From symmetry, \( \bar{x} = \bar{y} \).

\[
\bar{x} = \bar{y} \quad \text{and} \quad \frac{1}{2} \int \bar{x} = \frac{1}{2} \int_{0}^{1} x \left[ x^{\frac{1}{3}} - x^{3} \right] \, dx = \int_{0}^{1} x^{\frac{4}{3}} - x^{4} \, dx
\]

\[
= \left[ \frac{3}{7} x^{\frac{7}{3}} - \frac{1}{5} x^{5} \right]_{0}^{1} = \frac{3}{7} - \frac{1}{5} = \frac{8}{35}
\]

Then \( \bar{x} = \frac{16}{35} \).

\[
\bar{y} = \frac{16}{35} \quad \text{(same as } \bar{x})
\]

\[
\sqrt{\bar{x}} = 2 \pi \bar{x} A
\]

\[
\frac{16 \pi}{35} = \left( 2 \pi \bar{x} \right) \left( \frac{16}{35} \right)
\]

\[
\bar{x} = \frac{16}{35}
\]
4. (24) Simplify each of the following expressions.

a. (8) \( \ln \left( \frac{e^3}{e^5} \right) = \ln \left( e^{-2} \right) = -2 \)

b. (8) \( \lim_{n \to \infty} \left( \frac{n-4}{n-5} \right)^n = \lim_{n \to \infty} \left( \frac{n-4}{n} \right)^n = \lim_{n \to \infty} \left( \frac{1 + \frac{-4}{n}}{1 + \frac{-5}{n}} \right)^n \)

\( = \frac{e^{-4}}{e^{-5}} = e \)

c. (8) \( \ln 16 - \ln 4 = \ln \frac{16}{4} = \ln 4 \)

6. (16) A bank account is compounded continuously at an annual interest rate of 4%. How long does it take for the account to double in value?

\[ A(t) = A(0) e^{\frac{4}{100} t} \]

\[ 2A(0) = A(t) = A(0) e^{\frac{4}{100} t} \]

\[ 2 = e^{\frac{4t}{100}} \]

\[ \ln 2 = \frac{4t}{100} \]

\[ t = 25 \ln 2 \text{ years} \]
5. A boat is held by a bow line that passes through a ring mounted above the end of a dock. The level of the ring is 10 feet higher than the rope’s attachment point on the bow of the boat. If the boat is drifting away from the dock at the rate of 8 feet per minute, how fast is the line passing over the ring when the attachment point on the boat is exactly 26 feet from the ring?

\[ S^2 = x^2 + 100 \]

\[ 2S \frac{ds}{dt} = 2x \frac{dx}{dt} \]

\[ \frac{ds}{dt} = \frac{x}{S} \frac{dx}{dt} = \frac{24}{26}(8 \text{ ft/sec}) \]

When \( S = 26 \),

\[ x = \sqrt{(26)^2 - (10)^2} = 24 \]

Then \( \frac{ds}{dt} = \left(\frac{24}{26}\right)(8) \text{ ft/sec} = \frac{96}{13} \text{ ft/sec} \)
7. (18) Find and classify (as local maxima, local minima or neither) all of the critical points of the function \( f(x) = (x-1)^2(x+1)^2 \). Indicate where this function is increasing and where it is decreasing. Then find all inflection points for \( f \), and describe the concavity properties of its graph.

\[
f(x) = (x-1)^2(x+1)^2
\]

\[
f'(x) = 2(x-1)(x+1)^2 + 2(x+1)(x-1)^2
\]

\[
= (x-1)(x+1)(2(x+1) + 2(x-1))
\]

\[
= (x-1)(x+1)(4x) = 4x^3 - 4x
\]

\[
f''(x) = 12x^2 - 4
\]

\[
f'(x) = 0 \iff x = 1, -1, 0, \text{ and these are all the critical points.}
\]

At \( x = 0 \), \( f''(0) = -4 < 0 \), so \( x = 0 \) gives a local maximum \( f' \).

At \( x = \pm 1 \), \( f''(\pm 1) = 8 > 0 \), so \( x = \pm 1 \) gives local minima of \( f \).

The function is decreasing on \((-\infty, -1)\) and on \((0, 1)\).
The function is increasing on \((-1, 0)\) and on \((1, \infty)\).

\( f''(x) \) changes sign at \( x = \pm \sqrt{\frac{1}{3}} = \pm \frac{1}{\sqrt{3}} \), and these are the only inflection points.
The function is concave up on \((-\infty, -\frac{1}{\sqrt{3}})\) and concave down on \((-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})\).
8. (24) Compute each of the following:

a. \( \frac{2-2i}{3+4i} = \frac{2-2i}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{6-8-(6+8)i}{9+16} = -\frac{2}{25} - \frac{14}{25}i \)

b. \((1+i)^7 = \left( \sqrt{2} e^{\frac{\pi i}{4}} \right)^7 = 2^{7/2} e^{\frac{7\pi i}{4}} = \sqrt{128} \left( \cos \frac{7\pi}{4} + i \sin \frac{7\pi}{4} \right) = \sqrt{128} \left( \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right) = \sqrt{64} - i \sqrt{64} = 8 - 8i \)

c. (8) all three of the third roots of 27

\[ 27 = 27e^{\frac{i0 + i2\pi}{3}} = 27e^{\frac{i4\pi}{3}} \]

The three roots are \(3e^{\frac{i0}{3}}, 3e^{\frac{i2\pi}{3}}\) and \(3e^{\frac{i4\pi}{3}}\), i.e.,

\[ 3, 3\left( \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \text{ and } 3\left( \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right) \]

These are \(3, 3\left( \frac{1}{2} + i\frac{\sqrt{3}}{2} \right)\) and \(3\left( \frac{1}{2} - i\frac{\sqrt{3}}{2} \right)\), i.e. \(3, -\frac{3}{2} + \frac{3\sqrt{3}}{2}i, \) and \(-\frac{3}{2} - \frac{3\sqrt{3}}{2}i\)