Mathematics 1501 Hour Examination
W. L. Green
November 18, 2004

Directions: Do all problems. Show your work and justify your answers. Calculators are allowed, but this is a closed book examination. Please put your name and your recitation leader’s name on each page of your examination.

1 (40) Calculate each of the following derivatives and integrals.

\[ \frac{d}{dx} \left( \ln \left( \frac{3x}{x^2 + 1} \right) \right) = \frac{d}{dx} \left[ \ln 3 + \ln x - \ln(x^2 + 1) \right] \]

\[ = \frac{1}{x} - \frac{2x}{x^2 + 1} \]

---

b. \[ \int -\frac{3x}{1 + x^2} \, dx = \frac{3}{2} \int \frac{2x}{1 + x^2} \, dx = \ln (1 + x^2) + C \]

\[ u = 1 + x^2 \]
\[ du = 2x \, dx \]

---

c. \[ \frac{d}{dx} (2^{5x+1}) = \frac{d}{dx} \left[ e^{(5x+1) \ln 2} \right] = e^{(5x+1) \ln 2} \cdot 5 \ln 2 \]

\[ = (2^{5x+1}) (5 \ln 2) \]
1 (continued) Calculate each of the following derivatives and integrals.

d. \[ \int e^{x} \tan(e^{x}) \, dx = \int_{\pi/4}^{\pi/3} \tan u \, du \]
   \[ u = e^{x}, \quad du = e^{x} \, dx \]
   \[ = \left[ \ln |\sec x| \right]_{\pi/4}^{\pi/3} = \ln |\sec(\pi/3)| - \ln |\sec(\pi/4)| \]
   \[ = \ln 2 - \ln (\sqrt{2}) = \ln \left( \frac{2}{\sqrt{2}} \right) = \ln \sqrt{2} \]

e. \[ \int_{0}^{\pi/4} \sec x \, dx = \left[ \ln |\sec x + \tan x| \right]_{0}^{\pi/4} \]
   \[ = \ln |\sec(\pi/4) + \tan(\pi/4)| - \ln |\sec(0) + \tan(0)| \]
   \[ = \ln (\sqrt{2} + 1) - \ln 1 = \ln (\sqrt{2} + 1) \]

2. (12) Simplify each of the following. Make sure to give exact answers (e.g., use \( \sqrt{2} \) instead of 1.414, \( \frac{4}{3} \) or \( 1\frac{1}{3} \) instead of 1.33333); no credit will be given for decimal approximations.

a. \( \ln(4x) - \ln(4) = \ln 4 + \ln x - \ln 4 = \ln x \)

b. \( 2^{\log_{2}8} = 8 \)

c. \( \log_{3} \sqrt[3]{\frac{1}{243}} = \log_{3} \sqrt[3]{3^{-5}} = \log_{3} 3^{-5/2} = -\frac{5}{2} \)
3 (24) Let $R$ denote the region in the plane between the graph of $y = \cos x^2$ and the x-axis from $x = 0$ to $x = \frac{\sqrt{\pi}}{2}$.

a. Write out an integral, with limits, which represents the volume of the solid generated by revolving $R$ about the y-axis.

$$\int_0^{\sqrt{\frac{\pi}{2}}} 2\pi x \cos(x^2) \, dx$$

b. Evaluate the integral in part a of this problem.

$$\pi \int_0^{\sqrt{\frac{\pi}{2}}} 2x \cos(x^2) \, dx = \int_0^{\frac{\pi}{4}} \cos u \, du = \pi \left[ \sin u \right]_0^{\frac{\pi}{4}}$$

$$= \pi \left( \sin \left( \frac{\pi}{4} \right) - \sin 0 \right) = \pi \left( \frac{1}{\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}}$$
4. (24) A water tank is in the shape of a right circular cone with radius 2 and height 5. The cone is oriented along a vertical axis, with the point down. The tank is full of water.
a. How much water is in the tank?

\[ V = \frac{1}{3} \pi (2^2)(5) = \frac{20 \pi}{3} \]

\[ V = \int_0^5 \pi \left( -\frac{z}{5} x + 2 \right)^2 \, dx = \frac{20 \pi}{3} \]

b. How much work is done in pumping all of the water out of the tank? The weight density of water is about 62.5.

\[
\int_0^5 \left( 62.5 \right) \times \pi y^2 \, dx = \frac{125 \pi}{2} \int_0^5 \pi \left( 2 - \frac{2}{5} x \right)^2 \, dx
\]

\[
= \frac{125 \pi}{2} \int_0^5 \left( 4 x - \frac{8}{5} x^2 + \frac{4}{25} x^3 \right) \, dx =
\]

\[
= \frac{125 \pi}{2} \left[ 2x^2 - \frac{8}{15} x^3 + \frac{1}{25} x^4 \right]_0^5
\]

\[
= \frac{125 \pi}{2} \left[ 50 - \frac{8}{3} (25) + 25 \right] = \frac{3125 \pi}{6}
\]