1. (56) Calculate each of the following integrals and derivatives.

   a. \( \frac{d}{dx} [\tan^{-1}(3x)(\ln(3x))] \)
      
      \[ = \left( \frac{1}{1+9x^2} \right) \ln(3x) + (\tan^{-1}(3x)) \left( \frac{3}{3x} \right) \]
      
      \[ = \frac{\ln(3x)}{1+9x^2} + \frac{\tan^{-1}(3x)}{x} \]

   b. \( \int \frac{1}{\sqrt{25-x^2}} \, dx \)
      
      \[ = \frac{1}{5} \int \frac{dx}{\sqrt{1 - (\frac{x}{5})^2}} \]
      
      \[ = \int \frac{d\left( \frac{x}{5} \right)}{\sqrt{1 - (\frac{x}{5})^2}} = \tan^{-1}(\frac{x}{5}) + C \]

   c. \( \int xe^{4x} \, dx \)
      
      \[ = x \frac{e^{4x}}{4} - \int \frac{e^{4x}}{4} \, dx \]
      
      \[ = x \frac{e^{4x}}{4} - \frac{e^{4x}}{16} + C \]

   d. \( \int_0^{2\pi} \sin^2 x \, dx \)
      
      \[ = \int_0^{2\pi} \frac{1 - \cos 2x}{2} \, dx \]
      
      \[ = \left[ \frac{x}{2} - \frac{\sin 2x}{4} \right]_0^{2\pi} = \frac{2\pi}{2} = \pi \]
1. (continued) Calculate each of the following integrals and derivatives.

   e. \[ \frac{dx}{(x)(3+x)} \]
   
   \[ = \int \frac{1}{x} + \frac{-1/3}{3+x} \, dx \]
   
   \[ = \frac{1}{3} \ln |x| - \frac{1}{3} \ln |3+x| + C \]
   
   \[ = \frac{1}{3} \ln \left| \frac{x}{3+x} \right| + C \]

   f. \[ \frac{4x}{1+x^2} - \frac{2}{x\sqrt{x^2-1}} \, dx = 2 \int \frac{2x}{1+x^2} \, dx - 2 \int \frac{1}{x\sqrt{x^2-1}} \, dx \]
   
   \[ = 2 \ln(1+x^2) - 2 \arcsin x + C \]

   g. \[ \frac{d}{dx} \int_{0}^{x} \sec(t) \, dt = \sec x \]

2. (12) A spring has a natural length of 12 inches. It takes a force of 4 lb. to stretch it to a length of 18 inches. How much work is done in compressing it to a length of 8 inches?

   \[ F(x) = -kx \] and \[ -4 = -k\left(\frac{1}{2}\right), \] so \[ k = 8 \]

   \[ W = -\int_{0}^{-1/3} 8x \, dx = \left[4x^2\right]_{0}^{-1/3} = \frac{4}{9} \text{ ft} \cdot \text{lb} \]
3. (24)

a. (8) Find the area of the lens shaped region in the first quadrant between the graphs of \( y = x^3 \) and \( y = \sqrt[3]{x} \) and between \( x = 0 \) and \( x = 1 \).
\[
\int_0^1 \left( x^{\frac{1}{3}} - x^3 \right) \, dx = \left[ \frac{3}{4} x^{\frac{4}{3}} - \frac{x^4}{4} \right]_0^1 = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}
\]

b. (8) If this lens-shaped region is revolved around the y-axis, it sweeps out a solid. What is the volume of this solid?
\[
\int_0^1 2\pi x \left( x^{\frac{1}{3}} - x^3 \right) \, dx = 2\pi \int_0^1 x^{\frac{4}{3}} - x^4 \, dx
\]
\[
= 2\pi \left[ \frac{3}{7} x^{\frac{7}{3}} - \frac{x^5}{5} \right]_0^1 = 2\pi \left( \frac{3}{7} - \frac{1}{5} \right) = \frac{16\pi}{35}
\]

c. (8) Find the coordinates of the centroid of this lens-shaped region. \( \bar{y} = \bar{x} \) from symmetry.
\[
\bar{x} = \frac{1}{12} \int_0^1 x \left( x^{\frac{1}{3}} - x^3 \right) \, dx = 2\int_0^1 x^{\frac{4}{3}} - x^4 \, dx = 2\left[ \frac{3}{7} x^{\frac{7}{3}} - \frac{x^5}{5} \right]_0^1
\]
\[
= \frac{16}{35}
\]
4. (16) Show that among all rectangles with perimeter 16 inches, the one with the largest area is a square.

\[2(x + y) = 16\], so \[y = 8 - x\]

\[A(x) = xy = x(8-x) = 8x - x^2\]

\[A'(x) = 8 - 2x\], so \[A'(x) = 0 \iff x = 4\]

If \[x = 4\], then \[y = 8 - 4 = 4\], so this is a square.

Since \[A''(x) = -2\], this gives a maximum.

5. (16) A boat is held by a bow line that passes through a ring directly above the end of a dock. The level of the ring is 15 feet higher than the level of the bow of the boat. If the boat is drifting away from the dock along the waterline at the rate of 6 feet per minute, how fast is the line passing over the ring when the boat is exactly 36 feet from the dock?

\[s^2 = 15^2 + x^2\]

\[2s \frac{ds}{dt} = 2x \frac{dx}{dt}\]

When \[x = 36\], \[s = \sqrt{(15)^2 + (36)^2} = 39\]

Then \[2(39)\left(\frac{ds}{dt}\right) = 2(36)(6)\], so \[\frac{ds}{dt} = \frac{72}{13}\ ft/min\]
6. (48) Simplify each of the following expressions.

a. \( (8) \ln \sqrt{e} = \ln (e^{\frac{1}{2}}) = \frac{1}{2} \)

b. \( (8) \lim_{n \to \infty} \frac{(n-3)^n}{n+7} = \lim_{n \to \infty} \frac{(1 - \frac{3}{n})^n}{1 + \frac{7}{n}} = e^{-3} = e^{-10} \)

c. \( (8) \ln 16 - \ln 4 = \ln \left(\frac{16}{4}\right) = \ln 4 \)

d. \( (8) \sqrt{9 + 1} = \sqrt{10} \)

e. \( (8) \frac{2+2i}{3-i} = \frac{2+2i}{3-i} \left(\frac{3+i}{3+i}\right) = \frac{6+6i+2i-2}{9+1} = \)

\[ \frac{4+8i}{10} = \frac{2}{5} + \frac{4}{5}i \]

f. \( (8) (2-2i)^5 = \left[2 \sqrt{2} \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)\right]^5 = 2^\frac{5}{2} \left(e^{-\frac{\pi i}{4}}\right)^5 \)

\[ = 2^\frac{7}{2} e^{-i \frac{5\pi}{4}} = 128 \sqrt{2} (\cos(-\frac{5\pi}{4}) + i \sin(-\frac{5\pi}{4})) = -128 + 128i \]
7. (12) A certain radioactive isotope behaves in such a way that over the course of four days exactly one fifth of a given sample decays (i.e., turns into something else). What is the half-life of the original isotope?

\[
\frac{4}{5} y(t) = y(0) e^{kt}, \quad \text{so} \quad 4k = \ln \frac{4}{5}, \quad \text{so} \quad k = \frac{1}{4} \ln \left(\frac{4}{5}\right)
\]

\[
\frac{1}{2} y(t) = y(0) e^{kt} \quad \Rightarrow \quad -\ln 2 = \frac{1}{4} \ln \left(\frac{4}{5}\right)
\]

\[
\frac{-4 \ln 2}{\ln 4 - \ln 5} \quad \Rightarrow \quad t = \frac{4 \ln 2}{\ln 5 - \ln 4} \quad \text{days}
\]

8. (16) Find and classify (as local maxima, local minima or neither) all of the critical points of the function \( f(x) = x^3 - x^2 - x + 1 = (x-1)(x+1)(x+1) \). Indicate where this function is increasing and where it is decreasing. Then find all inflection points for \( f \), and describe the concavity properties of its graph.

\[
f'(x) = 2 (x-1)(x+1) + (x-1)^2 = (x-1)(2x^2 + 2x - 1) = (x-1)(3x+1)
\]

\[
f''(x) = 0 \quad \Rightarrow \quad x = 1 \text{ or } x = -\frac{1}{3}
\]

\[
f'(x) \leq 0 \quad \Rightarrow \quad -\frac{1}{3} \leq x \leq 1 \quad \text{and} \quad f'(x) \geq 0 \quad \Rightarrow \quad x \leq -\frac{1}{3} \text{ or } x \geq 1
\]

Thus \( f \) is increasing on \((-\infty, -\frac{1}{3}]\) and on \([1, \infty)\) and decreasing on \(\left[-\frac{1}{3}, 1\right] \). \(-\frac{1}{3}\) gives a local maximum, and 1 gives a local minimum.

\[
f''(x) = (3x+1) + (x-1)(3) = 6x - 2. \quad \text{Thus} \quad f''(x) \leq 0 \quad \Rightarrow \quad x \leq \frac{1}{3} \quad \text{and}
\]

\[
f''(x) \geq 0 \quad \Rightarrow \quad x \geq \frac{1}{3}. \quad \text{Thus} \ x = \frac{1}{3} \quad \text{gives an inflection point.}
\]

The graph is concave down on \((-\infty, \frac{1}{3}]\) and concave up on \(\left[\frac{1}{3}, \infty\right)\).