The existence of non-measurable sets

Theorem: Suppose $A$ is a measurable subset of $\mathbb{R}$, and suppose that every subset of $A$ is also measurable. Then $A$ has measure zero.

Proof: Let $E$ denote the set of all rational elements of $\mathbb{R}$. 

Choose a set $E \subseteq \mathbb{R}$ that consists of exactly one element from each coset of $E$. Then $E$ has the following properties:

1. $(E+r) \cap (E+r')$ is empty if $r \in E, r' \in E$ and $r \neq r'$
2. $E = \bigcup_{r \in E} E + r$

(That is, $\{E+r : r \in E\}$ is a partition of $\mathbb{R}$.)

If $x, z \in \mathbb{R}$, then $y + r = z + s$ for some $y, z \in E$, then $y - z = s - r \in \mathbb{Q}$, so that $y$ and $z$ lie in the same coset of $E$, say $y = z + s$, so $r = y - z$.

If $x, y \in E$ be the point in the coset $E + r$, and let $r = x - y$.

Then $x = y + r$ with $y \in E$, so $x \in E + r$.

Now, for each rational $r$, consider $A_r = A \cap (E + r)$. By hypothesis, each $A_r$ is measurable, and $A$ is the union of the $A_r$. Since $\{E + r : r \in E\}$ is a partition of $\mathbb{R}$, it suffices to show that $A_r$ has measure zero, and to show that $\bigcup_{r \in E} A_r = A$.

By inner regularity of Lebesgue measure, it suffices to show that for each closed subset $K$ of $E$, $\int_K 1_A \, d\lambda = 0$. Since each closed subset of $\mathbb{R}$ is a countable union of compact subsets, we may assume $K$ is compact. Let $H$ be the union of the translates $E + r$, where $r$ ranges over $E \cap [0, 1]$. Then $H$ is bounded.

If $K \subseteq E + r$, then $K + r \subseteq [x : |x| \leq MH]$ for each $r \in [0, 1]$. Then $|H| < \infty$.

Since $K \subseteq E + r$, property 1 above shows that the sets $E + r$ are pairwise disjoint. Thus $|H| = \sum |K + r| = \sum |K|$. Since $|H| < \infty$, we must have $|K| = 0$, as desired.

Corollary: If $A$ is a subset of $\mathbb{R}$ of positive measure, then $A$ contains a non-measurable set.


Note: One can show that if $E \subseteq \mathbb{R}$ and $|E| > 0$, then $\{x - y : x \in E, y \in E\}$ contains an interval of non-zero length centered at the origin. Thus, if $E$ is a subset of $\mathbb{R}$ with strictly positive measure, then $E$ contains a non-measurable set.