Mathematics 4317 Hour Examination  
October 26, 2006

Directions: Do all problems. Show your work, and justify your answers and assertions. This is a closed book examination, and calculators are allowed. Throughout this examination, the symbol “$\mathbb{R}$” will denote the real number system. Put all your answers in your blue book. Each part of each problem counts 12 points, so altogether there are 108 possible points. You have 50 minutes for this examination.

1. Let $C$ be a finite collection of compact connected subsets of $\mathbb{R}^p$ such that the intersection of any pair of sets in $C$ is non-empty.
   a) (12) Show that the union $U$ of all the sets in $C$ is connected.
   b) (12) Must the intersection of all of the sets in $C$ be non-empty? Give a proof or a counterexample.
   c) (12) Suppose $p = 1$. Show that there exist real numbers $a$ and $b$ such that $U = [a,b]$.

2. Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}^p$.
   a) (12) Let $s_n = \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$. Show that $\{s_n\}$ has a convergent subsequence.
   b) (12) Suppose that for each $n$, $\|x_{n+1}\| < \|x_n\|$. Must the sequence $\{x_n\}$ converge? Give a proof or a counterexample.
   c) (12) Suppose that $\limsup \frac{\|x_{n+1}\|}{\|x_n\|} < 1$. Show that $\{x_n\}$ converges to zero.

3. For each positive integer $n$ let $f_n$ be defined on the interval $[0,1]$ by $f_n(x) = \frac{nx}{nx + 1}$.
   a) (12) Show that the sequence $\{f_n\}$ converges pointwise on $[0,1]$, and compute the function $f$ to which it converges.
   b) (12) Is the convergence in part a) of this problem uniform on $[0,1]$? Why or why not?
   c) (12) Is the convergence in part a) of this problem uniform on $[1/2,1]$? Why or why not?
1a) Let A and B be open subsets of $\mathbb{R}^p$ such that $U = A \cup (B \cup U)$ and $(A \cup U) \cup (B \cup U)$ is empty. It suffices to show that $A \cup U$ or $B \cup U$ is empty. Now $U \subseteq A \cup B$, so for each $S$ in C, $S \subseteq U \subseteq A \cup B$. Thus $S = (S \cap A) \cup (S \cap B) = S$, and $(S \cap A) \cap (S \cap B) = (A \cap U) \cap (B \cup U) = \emptyset$. Since $S$ is connected, either $S \subseteq A$ or $S \subseteq B$, i.e. either $S \subseteq A$ or $S \subseteq B$. Thus each element of C is contained either in A or in B. We claim that all of the elements of C are contained in the same one of A and B. For suppose $S_1 \cap C$ and $S_2 \subseteq C$ with $S_1 \subseteq A$ and $S_2 \subseteq B$. Let $x \in S_1 \cap S_2$. Then $x \in S_1 \cap S_2 = (S_1 \cap A) \cap (S_2 \cap B) = (U \cap A) \cap (U \cap B) = \emptyset$, a contradiction. Thus either $U \subseteq A$ or $U \subseteq B$, so that either $U \cup B = \emptyset$ or $U \cup A = \emptyset$.

b) No. Let $s_1, s_2$, and $s_3$ be the sides of any non-degenerate triangle in $\mathbb{R}^2$.

c) By point a), $U$ is connected, so $U$ is an interval. Since $U$ is the union of finitely many connected sets, it is countable. A countable set must be closed and bounded, so $U$ must have the form $[a, b]$. 

2a) Suppose $\|x_n\| \leq M$ for all n. Then for all n, $\|x_{n+1} - x_n\| \leq \frac{1}{n} \|x_1 + \cdots + x_n\| \leq \frac{1}{n} (\|x_1\| + \cdots + \|x_n\|) = \frac{M}{n}$, so $\{x_n\}$ is also bounded. By the Bolzano-Weierstrass Theorem for sequences, $\{x_n\}$ has a convergent subsequence.

b) No. Let $x_n = (-1)^n (1 + \frac{1}{n})$ in $\mathbb{R}$, so that $|x_n| = 1 + \frac{1}{n} > |x_{n+1}| = 1 + \frac{1}{n+1}$ for all n and $|x_n| \leq 2$ for all n, but $\{x_n\}$ has subsequences converging to -1 and 1.

c) Let $\lim sup \frac{\|x_{n+1}\|}{\|x_n\|} = r < 1$. Then there exists $N$ such that for $n \geq N$ we have

\[
\frac{\|x_{n+1}\|}{\|x_n\|} \leq r < 1.
\]

Then for all $k \geq 0$ we have $\|x_{n+k}\| \leq r \|x_n\|$, so that $\|x_{n+k}\| \leq \|x_n\| r^k$ for all $k \geq 0$. Since $1 < r < 1$, $r^k \to 0$, so $x_n \to 0$.

3a) $\frac{x}{x+k+1} = 0$ for all n if $x = 0$. If $0 < x \leq 1$, then $\frac{x}{x+k+1} = \frac{x}{x \frac{1}{1+k}} \to 1$. Thus $f_n \to f$ pointwise on $[0, 1]$, where $f(x) = 0$ and $f(x) = 1$ for $0 < x \leq 1$.

b) No. If $x_n = \frac{1}{n}$, then $f_n(x_n) = \frac{1}{2}$. Thus $\|f_n - f\|_{[0, 1]} \geq \frac{1}{2}$, so $\|f_n - f\|_{[0, 1]} \to 0$.

c) Yes. For all $x$ in $[\frac{1}{2}, 1]$, $1 - \frac{nx}{nx+1} = \frac{nx+1-nx}{nx+1} = \frac{1}{nx+1} \leq \frac{1}{(\frac{1}{2})+1} = \frac{2}{n+2}$.

Thus $\|f - f_n\|_{[\frac{1}{2}, 1]} = \sup_{\frac{1}{2} \leq x \leq 1} |1 - \frac{nx}{nx+1}| \leq \frac{2}{n+2} \to 0$. 
