Mathematics 4431 Final Examination – December 15, 2006

Directions: Do all problems. Show your work, and justify your answers and assertions. (“Justify your answer” means give a proof or a counterexample.) This is a closed book examination, and calculators are allowed. Throughout this examination, the symbol “$\mathbb{R}$” will denote the real number system; unless otherwise specified, we will assume $\mathbb{R}$ has the usual topology. Please put your name on your bluebook.

1. (20) Let $X$ be a topological space, and let $p$ be a point of $X$.
a. Show that the intersection of any finite collection of neighborhoods of $p$ is again a neighborhood of $p$.
b. Give an example of a topological space $X$ and a point $p$ in $X$ such that closure of $\{p\}$ in $X$ is different from $\{p\}$ (i.e, such that $\{p\}$ is not a closed subset of $X$).

2. (10) Let $Q$ denote the set of all rational numbers, with the subspace (i.e., relative) topology from the usual topology on $\mathbb{R}$. Show that if $x$ is an element of $Q$, then the connected component of $Q$ that contains $x$ is $\{x\}$. [Hint: between any two distinct rationals in $\mathbb{R}$ there lies an irrational number.]

3. (10) Let $\{X_n\}$ be a collection (not necessarily countable) of compact Hausdorff spaces, and let $X$ be the product of the collection $\{X_n\}$, with the product topology. Show that a subset $S$ of $X$ is compact (in the relative topology) if and only if it is closed in $X$.

4. (20) Let $\{x_n\}$ be a sequence in a metric space $X$.
a. Show that if $p$ is a point of $X$, and if $\{x_n\}$ does not converge to $p$, then there exist a neighborhood $U$ of $p$ and a subsequence of $\{x_n\}$ that lies in the complement $X - U$ of $U$.
b. Suppose $X$ is compact, and suppose $\{x_n\}$ has exactly one cluster point in $X$. Show that $\{x_n\}$ converges.

5. (10) Suppose $f$ is a uniformly continuous function from a metric space $(X,d)$ into a complete metric space $(Y,\rho)$. Show that if $\{x_n\}$ is a Cauchy sequence in $(X,d)$, then $\{f(x_n)\}$ is a convergent sequence in $(Y,\rho)$.

6. (30) For each of the following, either give an example of what is described below or give a reason why no such example can exist:
a. a totally ordered set that is not well ordered
b. a compact pseudometric space that is not Hausdorff
c. a closed subset of $\mathbb{R}$ (with the usual topology) that has no limit points
d. a continuous function on a compact space $X$ into $\mathbb{R}$ that is not a closed function
e. a topological space with a countable base that is not separable
f. a net that is not a sequence.
1) Let $U_1, \ldots, U_n$ be neighborhoods of $p$. Choose open sets $O_1, \ldots, O_n$ with $p \in O_i \subseteq U_i$ for all $i$. Then $p \in \bigcap_{i=1}^n O_i \subseteq \bigcap_{i=1}^n U_i$, and $\bigcap_{i=1}^n O_i = \{p\}$.  

b) Let $X$ be any set with at least two points and the trivial topology. Then for each $p$ in $X$, the closure of $\{p\}$ is $X \neq \emptyset$.

2) Let $C$ be the component of $Q$ that contains $x$. Suppose $C$ contains two distinct points $y$ and $z$. We may assume $y < z$. Choose an interior number $v$ with $y < v < z$. Then $\{a \in Q : a < v\}$ and $\{b \in Q : v < b\}$ are non-empty open subsets of $Q$, and their intersections with $C$ disconnect $C$. $\{a \in Q : a < v\}$ and $\{b \in Q : v < b\}$ are open and closed and non-empty proper subsets of $C$.

3) Suppose $S$ is connected in $X$. The product of Hausdorff spaces is Hausdorff, so $S$ is a connected subset of a Hausdorff space $X$, so $S$ is closed in $X$. Conversely, if $S$ is closed in $X$, then $S$ is a closed subspace of the product of compact spaces, which is compact by the Tychonoff Theorem, so $S$ is itself compact.

4) a) Suppose $(x_n)$ does not converge to $p$. Then there exists a neighborhood $U$ of $p$ such that $(x_n)$ fails to lie eventually in $U$. Thus for each $m \geq 1$, there exists $n_m$ with $n_m \geq m$ and $x_{n_m} \notin U$. Choose $n_1 \geq 1$ with this property. Supposing that we have chosen $n_1, n_2, \ldots, n_m$ with $i \leq n_i$ for all $i$, we must choose $n_{m+1}$ and $x_{n_{m+1}} \notin U$ for all $i$, now choose $n_{m+1} \geq \max\{n_1, n_2, \ldots, n_m, m\}$ and $x_{n_{m+1}} \notin U$.

By induction, we get a subsequence $(x_{n_k})$. A $(x_{n_k})$ with $x_{n_k} \notin X-U$ for all $k$.

b) Let $\Phi$ be the unique cluster point of $(x_{n_k})$ in $X$. Suppose $(x_{n_k})$ does not converge to $\Phi$. Then by part a), there exists $n_k$ and $U$ with $x_{n_k} \notin X-U$ for all $i$. We may assume $U$ is an open neighborhood of $\Phi$. Thus $X-U$ is closed. Since $X$ is compact, $X-U$ is compact. Then $(x_{n_k})$ has a cluster point $x \in X-U$. But a cluster point of $(x_{n_k})$ is also a cluster point of $(x_n)$, so we have reached a contradiction.

5) Let $\varepsilon > 0$. Choose $M > 0$ such that $d(x, y) < \varepsilon$ implies $d(f(x_n), f(y_n)) < \varepsilon$. Now choose $N$ so that $n \geq N$ and $u \geq N$ imply that $d(x_n, x_m) < \varepsilon$. Then $d(f(x_n), f(x_m)) < \varepsilon$. Thus $\{f(x_n)\}$ is Cauchy. Since $(Y, \rho)$ is complete, $\{f(x_n)\}$ is convergent.
6 a) \( R \) with the usual ordering

6 b) Let \( X \) be any set with at least two elements and give it
the trivial pseudometric \( d(x, x) = 0 \) and \( d(x, y) = 1 \) for \( x \neq y \).

6 c) the empty set

6 d) This cannot happen. If \( C \) is closed in \( X \), then \( C \) is compact,
so \( f(C) \) is compact. Since \( R \) is Hausdorff, \( f(C) \) is closed in \( R \).

6 e) This cannot happen. Let \( \{B_i\} \) be a countable base for the
topology, and choose \( x_i \in B_i \). Then \( \{x_i\} \) is a countable dense
subset.

6 f) Let \( \mathcal{F} \) be the set of all finite subsets of \( R \). For each
\( F \in \mathcal{F} \), choose a point \( x \in R \). Then \( \{x \in R \mid x \in F \} \)
is a net in \( R \), where \( F_1 \prec F_2 \) means \( F_1 \subseteq F_2 \). Clearly \( F \neq \emptyset \).

Let \( \mathcal{J} \) be the collection of all open intervals
in \( R \) of the form \((-\epsilon, \epsilon)\), where \( \epsilon > 0 \). For each \( \epsilon \in \mathcal{J} \),
choose \( x \in X \). We order \( \mathcal{J} \) by \((-\epsilon, \epsilon) \prec (-\delta, \delta) \iff \delta < \epsilon.
Then \( \{x \} \in \mathcal{J} \) is a net in \( R \) and \( \{x \} \) converges to \( 0 \).
Note that \( \mathcal{J} \) is uncountable, so \( \mathcal{J} \neq \emptyset \).