Mathematics 1501 Hour Examination
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Directions: Do all problems. Show your work and justify your answers. This is a closed book examination, and calculators are not allowed. You are allowed one prepared sheet of material. Make sure your name and your recitation leader’s name are on all four pages of your examination.

1 (50) For each of the following functions \( f(x) \), calculate the derivative \( f'(x) \).

a. \( f(x) = (2x^3 - 3x^5)(1 + x + x^2) \)
\[
\begin{align*}
   f'(x) &= \left( 6x^2 - 15x^4 \right)(1 + x + x^2) + \\
        &+ \left( 2x^3 - 3x^5 \right)(1 + 2x)
\end{align*}
\]

b. \( f(x) = 7\sin x + 3\cos x \)
\[
f'(x) = 7 \cos x - 3 \sin x
\]

c. \( f(x) = \frac{2x + 3}{x^2 + 4} \)
\[
f'(x) = \frac{(2)(x^2 + 4) - (2x + 3)(2x)}{(x^2 + 4)^2}
\]
1 (continued) For each of the following functions \( f(x) \), calculate the derivative \( f'(x) \).

d. \( f(x) = x \sec x \)

\[
f'(x) = \sec x + x \sec x \tan x
\]

e. \( f(x) = \tan \sqrt{x^2 + 5} \)

\[
f'(x) = \left( \sec^2 \left( \sqrt{x^2 + 5} \right) \right) \left( \frac{1}{2 \sqrt{x^2 + 5}} \right) (2x)
\]

2. (10) Let \( f \) and \( g \) be differentiable functions that satisfy the following:

\[
\begin{align*}
  f(2) &= 5; \\
  g'(5) &= -3; \\
  f'(2) &= 7.
\end{align*}
\]

Let \( h \) be the composition function \( g \circ f \), so that \( h(x) = g(f(x)) \). What is the derivative of \( h \) at 2, and why?

\[
h'(2) = (g \circ f)'(2) = g'(f(2)) f'(2) = g'(5) f'(2) = (-3)(7) = -21
\]
3. (20) Consider the curve C given by the equation \( x^4y^3 + x^2y^4 = 2 \).

a. Find \( \frac{dy}{dx} \) at the point on C where \((x, y) = (-1, 1)\).

\[
\frac{d}{dx} \left( x^4 y^3 + x^2 y^4 \right) = \frac{d}{dx} (2) = 0
\]

\[
4x^3y^3 + 3x^4y^2 \frac{dy}{dx} + 2x^2y^4 + 4x^2y^3 \frac{dy}{dx} = 0
\]

\[
4x^3y^3 + 2x^2y^4 + (3x^4y^2 + 4x^2y^3) \frac{dy}{dx} = 0
\]

When \((x, y) = (-1, 1)\), we get

\[-4 - 2 + (3 + 4) \frac{dy}{dx} = 0\]

\[
\frac{dy}{dx} = \frac{6}{7} \quad \text{when} \quad (x, y) = (-1, 1)
\]

b. Find an equation for the line tangent to the curve C at the point where \((x, y) = (-1, 1)\).

\[
y - 1 = \frac{6}{7} (x + 1)
\]
4 (20) For each of the following sequences \( \{x_n\} \), decide whether the sequence has a limit or not. If it has a limit, find the limit.

a. \( x_n = \frac{2n^2 + n + 3}{5n^2 + 8} = \frac{2 + \frac{1}{n} + \frac{3}{n^2}}{5 + \frac{8}{n^2}} \to \frac{2}{5} \)

The limit exists and is \( \frac{2}{5} \).

b. \( x_n = \frac{5 \sin n}{n} \quad 0 \leq \left| \frac{5 \sin n}{n} - 0 \right| = 5 \left| \frac{\sin n}{n} \right| = \frac{5 \left| \sin n \right|}{n} \leq \frac{5}{n} \to 0 \)

The limit exists and is zero by the Squeezing Theorem.

c. \( x_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n} \to 1 - 0 \quad \sin \frac{1}{2^n} \to 0 \)

The limit exists and is 1.

d. \( x_n = \frac{5n^2 + 7}{n} = 5n + \frac{7}{n} \geq 5n \geq n \).

Since \( 5n \) (and \( \infty \)) are unbounded sequences, this limit does not exist.