
Directions: Do any five of the following eight problems. If you do more than five, tell me which five you want graded. Show your work, and justify your answers and assertions. Closed book, calculators are allowed. Throughout this examination, the symbol “$\mathbb{R}$” will denote the real number system, and $\| \|$ and $\cdot$ will denote the usual norm and inner product on $\mathbb{R}^p$.

1a. Let $A$ and $B$ be countably infinite sets. Show that $A \cup B$ is countably infinite.
b. Show that the set of all irrational numbers must be uncountable.

2. Let $D$ be the set of all irrational numbers.
a. Show that if $D$ is contained in $H$ and $H$ is closed, then $H = \mathbb{R}$.
b. Show that if a non-empty subset $F$ of $\mathbb{R}$ is closed and bounded, then the supremum and infimum of $F$ are elements of $F$.

3. Let $S$ be the subset of $\mathbb{R}$ given by $\{ \frac{1}{n} : n \text{ is a positive integer} \}$.
a. Show that every point of $S$ is a boundary point of $S$.
b. Show that the only cluster point of $S$ is zero.

4. For each of the following statements, either give a reason why the statement is true, or give a counterexample.
a. If $\{O_n\}$ is a nested collection of non-empty open sets, with $O_1 \supset O_2 \supset O_3 \supset \cdots$, then the intersection $\cap O_n$ of all the $\{O_n\}$ is not empty.
b. Every bounded subset of $\mathbb{R}$ is countable.
c. No finite subset of $\mathbb{R}^p$ can have a cluster point.
d. If the inner product $x \cdot y$ of two vectors $x$ and $y$ is zero, then either $\|x\| = 0$ or $\|y\| = 0$.

5. Let $K$ be a compact subset of $\mathbb{R}^p$. Let $\{x_n\}$ be a sequence in $K$.
a. Show that $\{x_n\}$ has a convergent subsequence.
b. Show that the limit of any convergent subsequence of $\{x_n\}$ must be an element of $K$.

6. Let $B$ be a closed ball in $\mathbb{R}^p$, and let $f$ be a linear function from $B$ into $\mathbb{R}$.
a. Show that the image $f(B)$ must be a closed bounded interval.
b. True or false, and give a reason: $f$ is uniformly continuous on $B$.

7. Let $f_n(x) = \frac{nx}{1 + (nx)^2}$. On the interval $[0, \infty)$ this sequence of functions converges pointwise to a function $f$. What is the function $f$? Is the convergence uniform on $[0, \infty)$? Is it uniform on the interval $[1, \infty)$? Give reasons for your answers.

8. Let $\{x_n\}$ be a sequence in $\mathbb{R}$, and suppose that the series $\sum x_n$ is convergent. Define two additional sequences by $x_n^+ = \frac{|x_n| + x_n}{2}$ and $x_n^- = \frac{|x_n| - x_n}{2}$.
a. Show that $x_n^+ + x_n^- = |x_n|$ and $x_n^+ - x_n^- = x_n$.
b. Show that $\sum x_n$ is absolutely convergent if and only if the series $\sum x_n^+$ and $\sum x_n^-$ are both convergent.
c. Show that $\sum x_n$ is conditionally convergent if and only if the series $\sum x_n^+$ and $\sum x_n^-$ are both divergent.
1a) Since $A \subseteq A \cup B$ and $A$ is infinite, $A \cup B$ is infinite. Let $A = \{a_1, a_2, a_3, \ldots\}$ be a list of the distinct elements of $A$. We define $f(a_i) = 2i$ for each $i = 1, 2, 3, \ldots$. Now either $B \setminus A$ is finite or $B \setminus A$ is infinite. If $B \setminus A$ is infinite, write $B = \{b_1, b_2, b_3, \ldots\}$ with the $b_i$ distinct. Then define $f(b_i) = 2i - 1$ for each $i$, and $f : A \cup B \to \mathbb{N}$ is a bijection. If $B \setminus A$ is finite, suppose $b_1, \ldots, b_k$ are its distinct elements. Then define $f(b_i) = 2i - 1$ for $i = 1, \ldots, k$. Then $f$ is a bijection of $A \cup B$ onto its image $f(A \cup B) \subseteq \mathbb{N}$.

Since $\mathbb{N}$ is countable, $f(A \cup B)$ is countable, so $A \cup B$ is countable.

b) $\mathbb{R}$ is the union of the rational numbers and the irrational numbers, and the rationals are countable. If the irrationals were countable, then by part a), $\mathbb{R}$ would be countable.

2a) Let $r \in \mathbb{R}$. Then for each $n \geq 1$, there exists an irrational $r_n$ with $|r - r_n| < \frac{1}{n}$. Thus there exists a sequence $(r_n)$ of irrationals with $r_n \to r$. Since $r_n \in D = H$ for all $n$ and $H$ is closed, $r = \lim_{n \to \infty} r_n \in H$. Thus $R \subseteq H$, so $R = H$.

b) Let $S = \sup F$. Then for each $n \geq 1$, there exists $s_n \in F$ with $s_n \leq S - \frac{1}{n} = S$. Then $s_n \to S$. Since $s_n \in F$ for all $n$ and $F$ is closed, $S = \lim_{n \to \infty} s_n \in F$. The proof for $\inf F$ is similar.

3a) Let $n \geq 1$. Let $U$ be a neighborhood of $\frac{1}{n}$. Choose $\varepsilon > 0$ so that $B = (\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon) \subseteq U$. Then $B$ contains $\frac{1}{n}$ (which lies in $S$), and $B$ contains some irrational number (which does not lie in $S$).

Thus $U$ contains a point of $S$ and a point of the complement of $S$.

d) If $c > 0$, the neighborhood $\left(\frac{c}{2}, +\infty\right)$ of $c$ contains only finitely many points of $S$. If $c < 0$, the neighborhood $(-\infty, 0)$ of $c$ contains no points of $S$. Thus in either case, there exist neighborhoods of $c$ that contain only finitely many points of $S$, so that $c$ is not a cluster point of $S$. 

4a) Counterexample: Let \( \Theta_n = (0, 1/n) \). Then \( \cap \Theta_n \) is empty.

4b) \([0, 1]\) is uncountable, by Cantor's diagonal argument.

4c) This is true: No neighborhood of \( x \) can contain infinitely many points of a finite set, no matter what \( x \) is.

4d) False: \( g \) and \( y \) are non-zero but orthogonal, \( x \cdot g = 0 \).

5a) Since \( K \) is compact, \( K \) is bounded. By the Bolzano-Weierstrass Theorem for sequences \( \{x_n\} \) has a convergent subsequence.

5b) Let \( \{x_{n_k}\} \) be a convergent subsequence of \( \{x_n\} \), and let \( x \) be its limit. Since \( K \) is closed and \( x_{n_k} \in K \) for all \( k \geq 1 \), \( x \in K \).

6a) The ball is compact and connected. Since every distance function \( D \) is continuous, \( A(B) \) is a compact connected subset of \( \mathbb{R}^n \).

6b) This is true: there exists \( M \) such that for all \( x \) only in \( TR^1 \), \( ||f(x) - f(y)|| \leq M ||x-y|| \), so \( f \) is Lipschitz on \( R^1 \).

6c) \( f \) is continuous and \( B \) is compact, so \( f \) is uniformly continuous on \( B \).

7a) \( f_n(x) = \frac{nx}{1+(nx)^2} \rightarrow 0 \) for every \( x \neq 0 \), and \( f_n(0) = 0 \)

for all \( n \). Thus \( f = 0 \). For any \( n \geq 1 \), \( f_n(\frac{1}{n}) = \frac{1}{1 + \frac{1}{n^2}} = 1 \), so

7b) \( \|f_n\| \geq \frac{1}{2} \). Since \( \|f_n\| \neq 0 \), \( f_n \) does not converge uniformly on \( [0, \infty) \). If \( x > 21 \), then \( |f_n(x) - 0| \leq \frac{nx}{1 + (nx)^2} \leq \frac{1}{|x|} \), \( \frac{1}{n} \rightarrow 0 \), so the convergence is uniform on \( [1, \infty) \).

8a) \( x_n^+ + x_n^- = \frac{1}{2} [x_n + x_n] + \frac{1}{2} [x_n - x_n] = \frac{1}{2} [x_n + x_n] = x_n \), and

8b) \( x_n^+ - x_n^- = \frac{1}{2} [x_n + x_n] - \frac{1}{2} [x_n - x_n] = x_n \), so

\[ \sum x_n^+ + \sum x_n^- = 2 \sum x_n \] and \( \sum x_n^+ - \sum x_n^- = \sum x_n \).
b) Suppose $\Sigma x_n^+$ and $\Sigma x_n^-$ are both convergent. Then $\Sigma |x_n| = \Sigma x_n^+ + \Sigma x_n^- \text{ is convergent. Suppose conversely that } \Sigma |x_n| $ is convergent. Then since $\Sigma x_n^+ \text{ is convergent}, \Sigma x_n^+ = \frac{1}{2} \Sigma |x_n| + \frac{1}{2} \Sigma x_n^- \text{ is convergent. Similarly, } \Sigma x_n^- = \frac{1}{2} \Sigma |x_n| - \frac{1}{2} \Sigma x_n^+ \text{ is convergent.}$

c) Observe that since $\Sigma x_n^+$ converges, and since $\Sigma x_n^+ - \Sigma x_n^- = \Sigma x_n$, either both of $\Sigma x_n^+$ and $\Sigma x_n^-$ must converge or both must diverge. If not both of $\Sigma x_n^+$ and $\Sigma x_n^-$ diverge, then by part b), $\Sigma x_n$ is not absolutely convergent; since it is convergent, it must then be conditionally convergent. Conversely, suppose that $\Sigma x_n$ is conditionally convergent. Then it is not absolutely convergent, so by part b), at least one of $\Sigma x_n^+$ and $\Sigma x_n^-$ must diverge. But then by the observation above, both $\Sigma x_n^+$ and $\Sigma x_n^-$ diverge.