Math 7334 – Supplement 1

Let $A$ be a Banach algebra with identity $I$.

**Def:** A linear functional on $A$ is *multiplicative* if $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$. It is a *character* of $A$ if it is multiplicative and non-zero.

**Note:** If $\varphi$ is multiplicative, then either $\varphi(I) = 1$ or $\varphi(I) = 0$. In the latter case $\varphi = 0$. Thus a multiplicative linear functional is non-zero, $\varphi(I) = 1$.

Let $\varphi$ be a character on $A$. Then $\ker \varphi$ is an ideal. Since the ideals in $A/\ker \varphi$ correspond 1-1 with those in $\mathbb{C} = \varphi(A)$, ker $\varphi$ is maximal. For each $a \in A$ and $\lambda \in \mathbb{C}$ we have $\varphi(a - \lambda I) = 0 \Leftrightarrow \varphi(a) = \lambda$, so $\varphi(a)$ is the *unique* scalar such that $a - \lambda I \in \ker \varphi$. In particular $\varphi(a) \in \text{sp}(a)$, so $|\varphi(a)| \leq \|a\|$. 

**Cor:** $\varphi$ is bounded and $\|\varphi\| = 1$.

Now let $A$ be commutative, so that $A/\mathcal{M} \cong \mathbb{C}$ for any maximal ideal $\mathcal{M}$.

**Prop:** Let $\mathcal{M}$ be a maximal ideal in $A$. Then $\exists$ isomorphism $\Phi_{\mathcal{M}}$ of $A/\mathcal{M}$ onto $\mathbb{C}$, and $\varphi_{\mathcal{M}} = \Phi_{\mathcal{M}} \circ Q_{\mathcal{M}}$ is a character with kernel $\mathcal{M}$, where $Q_{\mathcal{M}}$ is the quotient map of $A$ onto $A/\mathcal{M}$. Moreover, if $\psi$ is a character with kernel $\mathcal{M}$, then $\psi = \varphi_{\mathcal{M}}$.

**Lemma:** Let $\rho, \psi$ be linear functionals on $V$ with $\ker \psi \supseteq \ker \rho = V_0$. Then $\exists \mu \in \mathbb{C}$ with $\psi = \mu \rho$.

**Pf:** WLOG $\rho \neq 0$. Now $\rho_0 : x + V_0 \to \rho(x)$ is an isomorphism of $V/V_0$ onto $\mathbb{C}$ and $\rho = \rho_0 Q$, where $Q$ is the quotient map. Similarly $\psi_0 : x + V_0 \to \psi(x)$ is linear with $\psi = \psi_0 Q$. Now $V/V_0 \cong \mathbb{C}$, say $V/V_0 = \{\lambda(x_0 + V) : \lambda \in \mathbb{C}\}$. Then $\psi_0 = \mu \rho_0$, where $\mu = \frac{\psi_0(x_0 + V_0)}{\rho_0(x_0 + V_0)} = \frac{\psi(x_0)}{\rho(x_0)}$. Thus $\rho = \rho_0 Q = \mu \rho_0 Q = \mu \psi$ as desired.

**Pf of Prop:** Since $\psi(I) = \rho_{\mathcal{M}}(I)$, $\psi = \varphi_{\mathcal{M}}$.

In particular, there exist characters on any commutative Banach algebra $A$ with identity, since there exist maximal ideals of $A$. Let $\text{max}(A)$, $\mathcal{X}(A)$ be the sets of all maximal ideals and characters of such an $A$, resp. Then we have maps

$$
\mathcal{M} \to \varphi_{\mathcal{M}} \quad \varphi \to \ker \varphi
$$

$$
\text{max}(A) \to \mathcal{X}(A) \quad \mathcal{X}(A) \to \text{max}(A)
$$

Since every character $\psi$ has a maximal ideal for a kernel, and is the unique character with this kernel, we have $\psi = \varphi_{\ker \psi}$. Thus $\psi \to \ker \psi \to \psi_{\ker \psi}$ is the identity. Since $\varphi_{\mathcal{M}}$ is a character with kernel $\mathcal{M}$ (for every maximal ideal $\mathcal{M}$), $\mathcal{M} \to \varphi_{\mathcal{M}} \to \ker \varphi_{\mathcal{M}}$ is the identity. Thus the maps $\mathcal{M} \to \varphi_{\mathcal{M}}$ and $\varphi \to \ker \varphi$ are inverses of one another, and so $\text{max}(A)$ and $\mathcal{X}(A)$ are in 1-1 correspondence under those maps.