Mathematics 2401 Hour Examination
W. L. Green
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Directions: Do all problems. Show your work and justify your answers. This is a closed book examination, and calculators are allowed. Make sure your name is on all four pages of your examination.

1. (48) A circular plate occupies the space in the x-y-plane described by \( x^2 + y^2 \leq 25 \). Suppose the temperature at any point in the plate is given by the function \( f(x,y) = xy \).

a. Find the gradient of \( f \).

\[
\nabla f(x, y) = (y, x)
\]

b. Find the directional derivative of \( f \) at (2,3) in the direction of the unit vector \( u \) that points from (0,0) toward (1,-1).

\[
u = \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) \quad \text{and} \quad \nabla f(2, 3) = (3, 2)
\]

\[
(3, 2) \cdot \left( \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right) = \frac{1}{\sqrt{2}}
\]

c. In what direction is the temperature increasing most rapidly at the point (2,3)?

In the direction of the gradient \((3, 2)\)

d. A certain bug can only survive if the temperature under his feet is between 5 degrees and 7 degrees. Suppose we put him down on the plate at the point (2,3). Sketch the region on the plate that the bug can reach without risking his survival. [Hint: What kind of curves are the level curves of the function \( f \)?]
A circular plate occupies the space in the x-y-plane described by $x^2 + y^2 \leq 25$. Suppose the temperature at any point in the plate is given by the function $f(x, y) = xy$.

e. Another bug, more tolerant of temperature, runs in a straight line from the center of the plate to a point on the boundary. If he runs at a constant velocity of $(dx/dt, dy/dt) = (3, 4)$, how fast is the temperature changing under his feet at time $t$?

$$
\frac{d}{dt} (f(x, y)) = \nabla f(x, y) \cdot (\frac{dx}{dt}, \frac{dy}{dt}) = (y, x) \cdot (\frac{dx}{dt}, \frac{dy}{dt})
$$

$(x, y) = (3t, 4t)$, so the rate of change is $(4t, 3t) \cdot (3, 4) = 24t$.

$(x, y) = (3t, 4t)$, so $f(x, y) = (3t)(4t) = 12t^2$, which has derivative $24t$.

f. Show that the hottest points on the plate must be on the boundary of the plate. [Hint: You don't have to find these points to show that they must lie on the boundary.]

At an interior point of the plate, we have $\nabla f(x, y) = (0, 0)$ whenever $(x, y)$ is a local minimum or maximum. But

$(y, x) = (0, 0)$ only at the origin, where the 2nd derivative is

$$
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}.
$$

This is indefinite, so $(0, 0)$ gives a saddle point.

2. (36) Let $f(x, y) = x^4 + y^4 - 4xy + 1$.

a. Find equations for the tangent line and the normal line to the curve $f(x, y) = 7$ at the point $(1, -1)$.

$$
\nabla f(x, y) = (4x^3 - 4y, 4y^3 - 4x) \quad \text{and} \quad \nabla f(1, -1) = (8, -8)
$$

The tangent line has equation $(8, -8) \cdot (x - 1, y + 1) = 0$, i.e., $y = x - 2$.

The normal line has equation $(8, 8) \cdot (x - 1, y + 1) = 0$, i.e., $y = x$. 

2. (continued) Let \( f(x, y) = x^4 + y^4 - 4xy + 1 \).

b. Find the local maximum and minimum values and saddle points of the function \( f \).

\[
\nabla f(x, y) = (4x^3 - 4y, 4y^3 - 4x)
\]

\[
\nabla f(x, y) = (0, 0) \iff x^3 = y \quad \text{and} \quad y^3 = x
\]

These equations imply that \( x = y^3 = (x^3)^{\frac{1}{3}} = x \), i.e. that \( x^9 - x = 0 \), i.e. that \( x(x^8 - 1) = 0 \). This holds only if \( x = 0 \), \( x = 1 \) or \( x = -1 \).

The critical points are \((0, 0)\), \((1, 1)\) and \((-1, -1)\).

The 2nd derivative matrix is

\[
\begin{bmatrix}
8x^2 & -4 \\
-4 & 8y^2
\end{bmatrix}
\]

At \((0, 0)\), this matrix is

\[
\begin{bmatrix}
0 & -4 \\
-4 & 0
\end{bmatrix}
\]

which is indefinite.

\((0, 0)\) is a saddle point.

At \((1, 1)\) the matrix is

\[
\begin{bmatrix}
8 & -4 \\
-4 & 8
\end{bmatrix}
\]

which is positive definite, since \( 8 > 0 \) and \( \det \begin{bmatrix} 8 & -4 \\ -4 & 8 \end{bmatrix} > 0 \)

\(-1 = f(1, 1)\) is a local minimum value.

At \((-1, -1)\), the matrix is the same as at \((1, 1)\), so

\(-1 = f(-1, -1)\) is a local minimum value.

There are no local maximum values.
4 (20) Use the method of Lagrange multipliers to find the minimum and maximum values of the function \( x^2 + y^2 \) subject to the constraint \( x^4 + y^4 = 1 \).

\[
(2x, 2y) = \left( \frac{4}{2}, \frac{4}{2} \right) \iff
\frac{2x}{4} = 4 \frac{x^3}{x^4} \quad \iff \quad \frac{x}{2} = 4 \frac{x^3}{x^4} \quad \iff \quad \frac{x}{2} = 4 \frac{x^3}{x^4}
\]

Either \( x = 0 \) or \( x^2 = \frac{1}{2} \). (\( x = 0 \) means \( x = y = 0 \), so \( x^4 + y^4 = 0 \) is not valid.)

Either \( y = 0 \) or \( y^2 = \frac{1}{2} \).

If \( x \neq 0 \) and \( y \neq 0 \), then \( x^2 = y^2 \), so \( x^4 + y^4 = 1 \) gives \( 2x^2 = 1 \), i.e., \( x^2 = \frac{1}{\sqrt{2}} \). Then \( y^2 = x^2 = \frac{1}{\sqrt{2}} \).

This gives a value \( \sqrt{2} \) for \( x^2 + y^2 \).

If \( x = 0 \), then \( y^4 = 1 \), so \( y^2 = 1 \), so \( x^2 + y^2 = 1 \).

If \( y = 0 \), then \( x^4 = 1 \), so \( x^2 = 1 \), so \( x^2 + y^2 = 1 \).

The minimum value is 1, and the maximum value is \( \sqrt{2} \).