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Mathematics 2411 Hour Examination

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Directions: Do all problems. Show your work and justify your answers. This is a closed book examination, and calculators are allowed. Make sure your name is on all four pages of your examination. You will get four points for it.

1. (24) Let $f(x,y) = e^{xy} + xy + y^2$.

a. Find all the first order partial derivatives of f. Are these continuous functions? Why or why not.

$\frac{\partial f}{\partial x} = ye^{xy} + y \qquad \frac{\partial f}{\partial y} = xe^{xy} + x + 2y$

These are products, sums and compositions of exponential functions and polynomials, and so are continuous.

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b. Find all the second order partial derivatives of f. Make sure you specify which is which.

$\frac{\partial^2 f}{\partial x^2} = y^2 e^{xy} \qquad \frac{\partial^2 f}{\partial y \partial x} = e^{xy} + xye^{xy} + 1$

$\frac{\partial^2 f}{\partial x \partial y} = e^{xy} + xye^{xy} + 1 \qquad \frac{\partial^2 f}{\partial y^2} = x^2 e^{xy} + 2$

c. Give an example of a function $g(x,y)$ that is *not* continuous at the origin, but has a partial derivative with respect to x at the origin.

$g(x,y) = \frac{xy}{x^2+y^2}$ for $(x,y) \neq (0,0)$

and $g(0,0) = 0$ (See your text.)

2. (24) Let $\mathbf{c}(t) = (x(t), y(t), z(t))$ be a differentiable function, and suppose that for each t, $\mathbf{c}(t)$ lies on the surface $x^2 + y^2 + z^2 = 25$. Let C be the curve traced out by $\mathbf{c}(t)$.

a. Show that for each t, the tangent vector **T**(t) to the curve C at the point $\mathbf{c}(t)$ is perpendicular to the vector $\mathbf{c}(t)$.

$$0 = \frac{d}{dt}(25) = \frac{d}{dt}(x^2+y^2+z^2) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}$$

$$= 2(x, y, z)\cdot\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) = 2\underline{c}(t)\cdot\underline{T}(t)$$

Thus $\underline{c}(t)$ and $\underline{T}(t)$ are perpendicular.

b. If $\mathbf{c}(0) = (2\sqrt{2}, -2\sqrt{2}, 4)$, and $\mathbf{T}(0) = (2\sqrt{2}, 2\sqrt{2}, 0)$, find an equation for the tangent line to C at the point $\mathbf{c}(0)$.

$x(t) = 2\sqrt{2} + t(2\sqrt{2})$

$y(t) = -2\sqrt{2} + t(2\sqrt{2})$ or $\underline{x}(t) = \begin{bmatrix} 2\sqrt{2} \\ -2\sqrt{2} \\ 4 \end{bmatrix} + t\begin{bmatrix} 2\sqrt{2} \\ 2\sqrt{2} \\ 0 \end{bmatrix}$

$z(t) = 4$

c. If $\mathbf{c}(0) = (2\sqrt{2}, -2\sqrt{2}, 4)$, and $\mathbf{T}(0) = (2\sqrt{2}, 2\sqrt{2}, 0)$, find the directional derivative of the function $g(x,y,z) = x^2 + y^2 + z^2$ in the direction in which the tangent vector $\mathbf{T}(0)$ points.

The vector $\underline{T}(0)$ is parallel to the tangent plane to the surface $x^2+y^2+z^2 = 25$, and so is orthogonal to the gradient $(2x, 2y, 2z)$ at $\underline{c}(0)$. The directional derivative is therefore zero.

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3. (24)

a. Decide whether or not the limit $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2}$ exists or not. If it does exist, what is its value?

$$\left|\frac{x^3}{x^2+y^2} - 0\right| = \left|\frac{x^3}{x^2+y^2}\right| = |x|\left|\frac{x^2}{x^2+y^2}\right| \le |x|$$

as $(x,y)\to(0,0)$, $|x|\to 0$. Thus the limit exists and is zero.

b. Show that the intersection of two open subsets of the plane must also be an open set.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be open, and suppose $x \in \mathcal{O}_1 \cap \mathcal{O}_2$. Since $\mathcal{O}_1$ is open and $x \in \mathcal{O}_1$, there exists $\delta_1 > 0$ such that if $\|z-x\| < \delta_1$, then $z \in \mathcal{O}_1$. Since $\mathcal{O}_2$ is open and $x \in \mathcal{O}_2$, there exists $\delta_2 > 0$ such that if $\|z-x\| < \delta_2$, then $z \in \mathcal{O}_2$. Therefore if $\delta =$ minimum of δ_1 and δ_2, then $\delta > 0$ and for $\|z-x\| < \delta$ we have $z \in \mathcal{O}_1 \cap \mathcal{O}_2$. Thus $\mathcal{O}_1 \cap \mathcal{O}_2$ is open.

4. (24) Recall that the second remainder term $R_2(x_0,h)$ in Taylor's theorem is given by a formula of the form $R_2(x_0,h) = \sum_{i,j,k=1}^{m} \frac{1}{3!}\frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(\mathbf{c}_{ijk})h_i h_j h_k$.

a. Find the second order Taylor formula for the function $f(x,y) = \sin(xy)$ at $\mathbf{x}_0 = (0,0)$.

$f(0,0)=0$ and $\nabla f(0,0) = (y\cos xy,\ x\cos xy)\Big|_{(0,0)} = (0,0)$

The hessian at $(0,0)$ is given by $\frac{1}{2}\begin{bmatrix} -y^2\sin xy & -xy\sin xy + \cos xy \\ -xy\sin xy + \cos xy & -x^2\sin xy \end{bmatrix}$

evaluated at $(0,0)$, i.e. by $\frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Thus

$f(h_1,h_2) = 0 + (0,0)\cdot(h_1,h_2) + \frac{1}{2}\begin{bmatrix} h_1 & h_2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$ ~~[illegible]~~ $+ R_2(0,h)$

$= h_1 h_2 + R_2(\underline{0},\underline{h})$. The second order Taylor formula is thus $h_1 h_2$.

b. If $f(x,y) = a_1 + a_2x + a_3y + a_4x^2 + a_5y^2 + a_6xy$, where $a_1, a_2, \ldots, a_6$ are constants, then the remainder term $R(\mathbf{x}_0,\mathbf{h})$ for f is actually zero. Give a general reason why this should be so, based on the formulas for the remainder.

The remainder ~~term~~ is a sum of terms, each of which involves a 3rd order partial derivative, and all of the third order partial derivatives of the function f are everywhere zero. Thus $R_2(\underline{x}_0,\underline{h}) = 0$.