Mathematics 2411 Hour Examination
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April 10, 2007

Directions: Do all problems. Show your work and justify your answers. This is a closed book examination, and calculators are allowed. Make sure your name is on all four pages of your examination. You will get four points for it.

1. (24) Let D be the quarter disc in the first quadrant of the x-y-plane with boundary given by x = 0, y = 0, and x^2 + y^2 = 4.
   a. (6) By computing a double integral, verify that the area of D is \pi.

\[
\iint_D dA = \int_0^{\pi/2} \int_0^2 r \, dr \, d\theta = \int_0^{\pi/2} \left[ \frac{r^2}{2} \right]_0^2 \, d\theta
\]

\[= \int_0^{\pi/2} 2 \, d\theta = 2 \left( \frac{\pi}{2} \right) = \pi \]

b. (12) Find the coordinates of the center of mass of D, assuming that the density is constant throughout D.

\[
\left( \bar{x}, \bar{y} \right) = \left( \int_0^{\pi/2} \int_0^2 \left( x \cdot c_{x \cdot} \, d\theta \right) \, r \, dr \, d\theta \right) \left( \int_0^{\pi/2} \int_0^2 \, r^2 \, d\theta \, dr \right)
\]

\[= \left( \frac{2^3}{3} \right) = \frac{8}{3 \pi} \]  
so  \[ \bar{x} = \frac{8}{3 \pi} \]  
By symmetry  \[ \bar{y} = \bar{x} = \frac{4}{5 \pi} \]

c. (6) Find the area of the region R in the first quadrant of the x-y-plane with boundary given by x = 0, y = 0, and 4x^2 + y^2 = 4. Let u = 2x and v = y, so that 4x^2 + y^2 = 4 becomes \( u^2 + v^2 = 4 \), \( x = 0 \) becomes \( u = 0 \), and \( y = 0 \) becomes \( v = 0 \). Then

\[
\iint_R 1 \, dA = \iiint_0^2 1 \, dA = 2 \iiint_0^2 1 \, dA = 2 \text{ (area of R)}\]

so the area of R is \( \frac{8}{\pi} \).
2. (24) Ice cream is packed into the space above the x-y-plane between the cone $x^2 + y^2 = z^2$ and the sphere $x^2 + y^2 + z^2 = 1$. (This is a more or less normal shape for an ice cream cone.)

a. What is the volume of the ice cream in this space?

$$\int_{0}^{\pi/4} \int_{0}^{1} \int_{0}^{1} \rho^2 \sin \phi \ d\rho \ d\phi \ d\theta =$$

$$= 2\pi \left( \int_{0}^{\pi/4} \sin \phi \ d\phi \right) \left( \int_{0}^{1} \rho^2 \ d\rho \right) =$$

$$= 2\pi \left[ -\cos \phi \right]_{0}^{\pi/4} \left( \frac{1}{3} \right) = \frac{2\pi}{3} \left( 1 - \frac{1}{\sqrt{2}} \right)$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \phi \leq \frac{\pi}{4}$$

$$0 \leq \rho \leq 1$$

b. The ice cream is allowed to melt, and over time all of the ice cream above the top of the cone runs off. What is the volume of ice cream that remains in the cone. [Hint: the top of the cone lies at a height $z$ that has to satisfy both $x^2 + y^2 = z^2$ and $x^2 + y^2 + z^2 = 1$.]

$$\int_{0}^{\pi/4} \int_{0}^{\sqrt{z^2}} \int_{0}^{\sqrt{z^2}} dz \ d\rho \ dr \ d\theta =$$

$$= \int_{0}^{\pi/4} \int_{0}^{\sqrt{z^2}} \left( \frac{r^2}{2} - r \right) r \ dr \ d\theta =$$

$$= \int_{0}^{\pi/4} \left[ \frac{r^3}{3} - \frac{r^2}{2} \right] \frac{1}{\sqrt{2}} \ d\theta =$$

$$= \left[ \frac{1}{3} \sqrt{2} - \frac{1}{6} \right] \frac{1}{2} = \frac{\pi}{6\sqrt{2}}$$
3. (24) Let \( T \) be the triangle in the plane with vertices at \((0,0), (0,1)\) and \((1,0)\), oriented counterclockwise (from \((0,0)\) to \((1,0)\) to \((0,1)\)).

a. Find the line integral \( \int_T \mathbf{F} \cdot d\mathbf{s} \), where \( \mathbf{F}(x,y) = (2x, 0) \).

\[ T_1 : (t,0), \quad 0 \leq t \leq 1 \]
\[ T_2 : (1-t)(0,1) + t \,(0,1) = (1-t, t), \quad 0 \leq t \leq 1 \]
\[ T_3 : (0,1-t), \quad 0 \leq t \leq 1 \]

\[ T_1 : \int_0^1 (2t,0) \cdot (1,0) \, dt = \int_0^1 2t \, dt = 1 \]
\[ T_2 : \int_0^1 (2(1-t),0) \cdot (-1,1) \, dt = \int_0^1 2t - 2 \, dt = \left[ t^2 - 2t \right]_0^1 = -1 \]
\[ T_3 : \int_0^1 (0,0) \cdot (0,-1) \, dt = 0 \]

The sum of these is zero, so \( \int_T \mathbf{F} \cdot d\mathbf{s} = 0 \).

b. Find the line integral \( \int_T \mathbf{F} \cdot d\mathbf{s} \), where \( \mathbf{F}(x,y) = (2y, 2x) \).

This line integral is zero, since \( T \) is a closed curve and \( \mathbf{F} \) is the gradient of \( 2xy \) throughout the plane.
4. (24) Let \( S \) be the surface given by \((x, y, z) = (r \cos \theta, r \sin \theta, 2r)\), where \(0 \leq r\) and \(0 \leq \theta \leq 2\pi\).

a. Where precisely is this surface regular?

\[
T_r = (\cos \theta, \sin \theta, 2) \quad \text{and} \quad T_\theta = (-r \sin \theta, r \cos \theta, 0), \quad \text{so}
\]

\[
T_r \times T_\theta = \det \begin{bmatrix}
\cos \theta & \sin \theta & 2 \\
-r \sin \theta & r \cos \theta & 0 \\
1 & 0 & 0
\end{bmatrix} = (-2r \cos \theta, -2r \sin \theta, r^2 \cos^2 \theta + r^2 \sin^2 \theta)
\]

\[
= r (-2 \cos \theta, -2 \sin \theta, 1), \quad \text{Thus} \quad \| T_r \times T_\theta \| = r \sqrt{4 \cos^2 \theta + 4 \sin^2 \theta + 1}
\]

\[
= r \sqrt{5}. \quad \text{This length is zero if and only if} \ r = 0, \ \text{so the surface is regular everywhere except at} \ (0, 0, 0).
\]

b. Find the tangent plane to the surface \( S \) at the point \((-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2)\).

At \((-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 2)\) we have \(r = 1\) and \(\cos \theta = -\frac{1}{\sqrt{2}}, \ \sin \theta = \frac{1}{\sqrt{2}}\).

Thus for this point we have \(r = 1\) and \(\theta = \frac{3\pi}{4}\). Then

\[
T_r \times T_\theta = r (-2 \cos \theta, -2 \sin \theta, 1) = (-2 \cos \left(-\frac{1}{\sqrt{2}}\right), -2 \left(\frac{1}{\sqrt{2}}\right), 1)
\]

\[
= (\sqrt{2}, -\sqrt{2}, 1)
\]

The tangent plane has equation \((\sqrt{2}, -\sqrt{2}, 1) \cdot \left(x + \frac{1}{\sqrt{2}}, y - \frac{1}{\sqrt{2}}, z - 2\right) = 0\)

i.e. \(\sqrt{2}x - \sqrt{2}y + z = 0\)