Def: A collection τ of subsets of a set X is a *topology* for X if

a) $\phi, X \in \tau$,

b) the intersection of any two elements of τ is an element of τ, and

c) the union of any subcollection of τ is an element of τ.

Def: A set X together with a topology for X is called a *topological space.*

Def: If (X,τ) is a topological space, $X \supseteq A$, and $p \in X$, then p is an *accumulation point of* A if every $U \in \tau$ with $p \in U$ meets A in a point other than p, i.e., if $U \in \tau$ and $p \in U$ imply $U \cap A \not\subseteq \{p\}$.

Def: The set A' of all accumulation points of the set A is called the *derived set of* A. The set $A \cup A'$ is called *the closure of* A and is denoted by cl A or $\bar{A}$.

Def: If $A = \text{cl } A$, then A is called a *closed set.*

Prop. 1.1: A set A is closed if and only if $A' \subseteq A$.

Pf: If $A' \subseteq A$, then $\text{cl}(A) = A' \cup A \subseteq A \cup A = A \subseteq A' \cup A = \text{cl}(A)$, so $A = \text{cl}\, A$. (The converse is trivial).

Def: A set A is *open* if its complement $X - A (= \{x \in X : x \notin A\})$ is closed.

Thm. 1.2: If (X,τ) is a topological space, a set $U \subseteq X$ is open if and only if $U \in \tau$.

Pf: Let $C = X - U = \{x \in X : x \notin U\}$. Then U is open $\Leftrightarrow$ C is closed $\Leftrightarrow$ $C' \subseteq C$. Suppose first $U \in \tau$. Let $p \in C'$. We must show $p \in C$. If not, then $p \in U$. But then (since $p \in C'$) U meets $C = X - U$ (in a point other than p, even!), a contradiction. Suppose conversely that U is open, i.e., that $C' \subseteq C$. We must show that $U \in \tau$. Let $q \in U$. (If U is empty, then $U \in \tau$.) Since $C' \subseteq C = X - U$, $q \notin C'$. By definition of C' and the fact that $q \notin C'$, there exists $V_q \in \tau$ with $q \in V_q$ and such that $V_q \cap C \subseteq \{q\}$. Since $q \notin C$, we have in fact that $V_q \cap C = \phi$, i.e., $V_q \subseteq X - C = U$. Since $q \in U$ is arbitrary, we have $U = \bigcup\limits_{q \in U} \{q\} \subseteq \bigcup\limits_{q \in U} V_q \subseteq U$, so $U = \bigcup\limits_{q \in U} V_q$. Since each $V_q \in \tau$, the union U lies in τ.

Def: A set N is a *neighborhood of the point* p if there is an open set U such that $p \in U \subseteq N$. A set N is a *neighborhood of the set* S is it is a neighborhood of each point of S.

Remark: If N is open, then clearly N is a neighborhood of each of its points (take $U = N$). If N is a neighborhood of p, and $N \subseteq M$, then clearly M is a neighborhood of p.

Def: The *interior* of a set S is the set of all p such that S is a neighborhood of p. We denote the interior of S by $\operatorname{int} S$.

Thm: 1.3 A point p is a member of $\operatorname{cl} S$ if and only if every neighborhood of p meets S.

Proof: Suppose every neighborhood of p meets S, we need to show that $p \in \operatorname{cl}(S)$. Since $S \subseteq \operatorname{cl} S$, we are done if $p \in S$. So assume $p \notin S$. Let $U \in \tau$ with $p \in U$. Then U is open, so U is a neighborhood of p, so U meets S. Since U is arbitrary subject to $U \in \tau$ and $p \in U$, we have $p \in S'$. Thus $p \in \operatorname{cl} S$. Conversely, suppose $p \in \operatorname{cl}(S)$. Let N be a neighborhood of p, and choose $U \in \tau$ with $p \in U$ and $U \subseteq N$. If $p \in S$, then $p \in N \cap S$, so N meets S. If $p \notin S$, then $p \in S'$, so U meets S (in a point other than p). In particular $N \cap S \neq \phi$.

Thm. 1.4: A set is open if and only if it is a neighborhood of each of its points.

Proof: If S is a neighborhood of each of its points, then for each $x \in S$, choose $V_x \in \tau$ with $x \in V_x \subseteq S$. Then $S = \bigcup_{x \in S} \{x\} \subseteq \bigcup_{x \in S} V_x \subseteq S$, so S is a union of open sets. The converse is obvious.

Thm. 1.5: If (X, τ) is a topological space, then

a) ϕ and X are closed sets,

b) the union of two closed sets is closed, and

c) the intersection of any collection of closed sets is closed.

Proof: Since $U \in \tau \Leftrightarrow U$ is open $\Leftrightarrow X - U$ is closed, this follows immediately from deMorgan's Laws.

Thm. 1.6: The closure of a set is a closed set. (That is, $\operatorname{cl}(\operatorname{cl}(A)) = \operatorname{cl}(A)$.)

Proof: $p \notin \operatorname{cl} A \Leftrightarrow$ some neighborhood of p is disjoint from $A \Leftrightarrow$ some neighborhood of p is contained in $X \backslash A \Leftrightarrow X \backslash A$ is a neighborhood of p, so $X \backslash \operatorname{cl} A = \operatorname{int}(X \backslash A)$. The theorem now follows from:

Thm. 1.7: The interior of a set is an open set.

Proof: If $p \in \operatorname{int} A$, then there exists an open U_p with $p \in U_p \subseteq A$. Clearly, A is a neighborhood of each point of U_p, so $U_p \subseteq \operatorname{int} A$. Thus $\operatorname{int} A = \bigcup_{p \in \operatorname{int} A} \{p\} \subseteq \bigcup_{p \in \operatorname{int} A} U_p \subseteq \operatorname{int} A$. Thus $\operatorname{int} A$ is a union of open sets and hence is open.

Thm. 1.8: If $A \subseteq B$, then $\operatorname{cl} A \subseteq \operatorname{cl} B$.

Proof: Let $A \subseteq B$. Then $p \in \operatorname{cl} A \Leftrightarrow$ every neighborhood of p meets $A \Rightarrow$ every neighborhood of p meets $B \Leftrightarrow p \in \operatorname{cl} B$.

Thm. 1.9: If $A \subseteq F$ and F is closed, then $\operatorname{cl} A \subseteq F$. (Immediate from Thm. 1.8, since $\operatorname{cl} F = F$ if F is closed.)

Thm. 1.10: If $U \subseteq A$ and U is open, then $U \subseteq \operatorname{int} A$.

Proof: If $p \in U \subseteq A$ and U is open, then A is a neighborhood of p, so $p \in \operatorname{int} A$.

Thm. 1.11: $\operatorname{cl} A = \bigcap\{F : A \subseteq F, F \text{ is closed}\}$ and $\operatorname{int} A = \bigcup\{U : U \subseteq A, U \text{ is open}\}$.

Proof: By Theorem 1.7, $\operatorname{int} A \subseteq \bigcup\{U : U \subseteq A, U \text{ open}\}$. By Thm. 1.10 this union is a subset of $\operatorname{int} A$ (since each member is a subset of $\operatorname{int} A$). By Thm. 1.9, $\operatorname{cl} A \subseteq \bigcap\{F : A \subseteq F, F \text{ is closed}\}$. But $\bigcap\{F : A \subseteq F, F \text{ is closed}\} \subseteq \operatorname{cl} A$, since $A \subseteq \operatorname{cl} A$ and $\operatorname{cl}(A)$ is closed.

Examples 1.12: Suppose X is a set. Each of the following determines a topology τ for X. (Exercise.)

a) The collection of all subsets of X (the "discrete topology")

b) $\tau = \{\phi, X\}$ (the "trivial topology")

c) $\tau = \{U \subseteq X : X - U \text{ is finite}\} \cup \{\phi\}$ (the "cofinite topology")

d) $\tau = \{U \subseteq X : X - U \text{ is countable }\} \cup \{\phi\}$ (the "cocountable topology")

e) Suppose $A \subseteq X$ is fixed. Then $\tau = \{X, A, \phi\}$ is a topology.

Def: Let (X, τ) be a topological space. A subset $\mathcal{B}$ of τ is a *base*, or *basis*, for τ if each element of τ is a union of elements of $\mathcal{B}$. [Example: All open balls in $\mathbb{R}^n$.]

Prop 1.13: Let $\mathcal{B}$ be a base for τ. Then a point p is an accumulation point of a set S if and only if every member of $\mathcal{B}$ containing p meets S in a point other than p.

Proof: $\Leftarrow$: Let $U \in \tau$ with $p \in U$. Then U is a union of elements of $\mathcal{B}$, so p lies in some $B \in \mathcal{B}$ with $B \subseteq U$. By hypothesis, B meets S in a point other than p, so the same is true of U (since $B \cap S \subseteq U \cap S$). The converse is trivial since $\mathcal{B} \subseteq \tau$.

Cor. 1.14: A point p is a member of $\operatorname{cl} S$ if and only if every element of $\mathcal{B}$ containing p meets S.

Proof: Let $p \in \operatorname{cl} S$. If $p \in S$, then for each $B \in \mathcal{B}$ with $p \in B$, $B \cap S \supseteq \{p\}$. If $p \notin S$, then $p \in S'$. By the proposition, each $B \in \mathcal{B}$ with $p \in B$ meets S (in a point other than p). Conversely, suppose each member B of $\mathcal{B}$ containing p meets S. If $p \in S$, then $p \in \operatorname{cl}(S)$, so we may assume $p \notin S$. Then each such B meets S (by hypothesis), and hence meets S

in a point other than p (since $p \notin S$). Hence $p \in S' \subseteq \operatorname{cl} S$.

Thm. 1.15: Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X such that

a) $X = \cup\mathcal{B}$,

b) if $B_1, B_2 \in \mathcal{B}$ and $p \in B_1 \cap B_2$, then there exists $B_3 \in \mathcal{B}$ with $p \in B_3 \subseteq B_1 \cap B_2$.

Then the collection τ of all unions of elements of $\mathcal{B}$ is a topology for X.

Proof: Clearly $\phi(= \cup\phi)$ and $X(= \cup\mathcal{B})$ are in τ. Also clearly, any union of unions of elements of $\mathcal{B}$ is an element of τ, so τ is closed under arbitrary unions. Let $U = \cup\mathcal{C}$, $V = \cup\mathcal{D}$, where $\mathcal{C} \subseteq \mathcal{B}$, $\mathcal{D} \subseteq \mathcal{B}$. Then, $U \cap V = (\cup\mathcal{C}) \cap (\cup\mathcal{D}) = \cup\{C \cap D : C \in \mathcal{C}, D \in \mathcal{D}\}$. ($x \in (\cup\mathcal{C}) \cap (\cup\mathcal{D}) \Leftrightarrow x \in \cup\mathcal{C}$ and $x \in \cup\mathcal{D} \Leftrightarrow x \in C$ and $x \in D$ for some $C \in \mathcal{C}, D \in \mathcal{D} \Leftrightarrow x \in C \cap D$ for some $C \in \mathcal{C}, D \in \mathcal{D}$.) Thus we need only show that for $C \in \mathcal{B}$ and $D \in \mathcal{B}$ the set $C \cap D$ is a union of elements of $\mathcal{B}$. For each $x \in C \cap D$, let $C_x \in \mathcal{B}$ be such that $x \in C_x \subseteq C \cap D$. Then $C \cap D = \bigcup\{C_x : x \in C \cap D\}$ (since $C \cap D = \bigcup\limits_{x \in C\cap D} \{x\} \subseteq \bigcup\limits_{x \in C \cap D} C_x \subseteq C \cap D$).

Note: If $\mathcal{B}$ is a base for a topology, then this topology is clearly unique.

Def: Suppose X is a set. A real valued function $\rho : X \times X \to \mathbb{R}$ is a *pseudometric*, or *semimetric*, if

a) $\rho(x, y) \geq 0$ for all x, y

b) $\rho(x, y) = \rho(y, x)$ for all x, y

c) $\rho(x, x) = 0$ for all x

d) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all x, y, z.

The pair (X, ρ) is called a *pseudometric space*.

Def: Let (X, ρ) be a pseudometric space, and let $x \in X$, $r > 0$ in $\mathbb{R}$. The set $C(x; r) = \{y \in X : \rho(x, y) < r\}$ is the *cell of radius r centered at x* (sometimes "ball" instead of "cell").

Thm. 1.16: Let (X, ρ) be a pseudometric space. Then the collection $\mathcal{B}$ of all cells is a base for a topology.

Proof: If $r_1, r_2 > 0$ in $\mathbb{R}$, $x_1, x_2 \in X$, consider $p \in C(x_1; r_1) \cap C(x_2, r_2)$. Choose r such that $0 < r < r_1 - \rho(x_1, p)$ and $0 < r < r_2 - \rho(x_2, p)$. (By definition of $C(x_i, r_i)$ $r_i > \rho(x_i, p)$.) Consider $z \in C(p, r)$. We have

$$\rho(x_1, z) \leq \rho(x_1, p) + \rho(p, z) < \rho(x_1, p) + r \leq \rho(x_1, p) + r_1 - \rho(x_1, p) = r_1,$$

so $z \in C(x_1; r_1)$. Similarly $z \in C(x_2; r_2)$. Thus $C(p; r) \subseteq C(x_1; r_1) \cap C(x_2; r_2)$. Clearly X is the union of all the cells in X. Now use Thm. 1.15.

Def: The topology in the preceding theorem is the topology *generated* by the pseudometric ρ. Two pseudometrics are *equivalent* if they generate the same topology.

Examples 1.17: Let X be a set. Then $\rho(x, y) = 0$ for all $x, y \in X$, and $\rho(x, y) = \begin{Bmatrix} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{Bmatrix}$ define pseudometrics on X.

Examples 1.18: On $X = \mathbb{R}^n$, ρ, r and d are pseudometrics, where

$$\rho(x, y) = \left[\sum_1^n (x_i - y_i)^2 \right]^{1/2}, \quad r(x, y) = \sum_1^n |x_i - y_i|, \quad d(x, y) = \max_i \{ |x_i - y_i| : 1 \leq i \leq n \}.$$

Moreover, these three pseudometrics are equivalent. (The topology they generate is the "usual" topology in $\mathbb{R}^n$.) See Math 4317.

Prop. 1.19: Suppose (X, τ) is a topological space and $A \subseteq X$. Then the collection $\tau_A = \{U \cap A : U \in \tau\}$ is a topology for A. (The topological space (A, τ_A) is called a *subspace* of (X, τ) and τ_A is called the *subspace topology*, or the *relative topology*, for A from τ.)

Proof: Since $A = X \cap A$, $\phi = \phi \cap A$, $(U_1 \cap A) \cap (U_2 \cap A) = (U_1 \cap U_2) \cap A$, and $\bigcup\{U \cap A : U \in \Gamma\} = \{\bigcup\{U : U \in \Gamma\}\} \cap A$, the result is immediate.

Prop. 1.20: Let (A, τ_A) be a subspace of a topological space (X, τ). Then $F \subseteq A$ is closed with respect to $\tau_A \Leftrightarrow$ there is a τ-closed subset S of X such that $F = S \cap A$.

Proof: Since $F \subseteq A$, clearly A is the disjoint union of F and $A \backslash F$, so $A \backslash F$ and F are complements of one another in A. Let T be any subset of X and write T^c and A^c for $X \backslash T$ and $X \backslash A$ respectively. Then $A \backslash (T \cap A) = A \cap (T \cap A)^c = A \cap (T^c \cup A^c) = (A \cap T^c) \cup (A \cap A^c) = A \cap T^c$. In particular, if $F = S \cap A$ with S τ-closed, then $A \backslash F = A \cap S^c$ with S^c τ-open. Since F and $A \backslash F$ are complements in A, this says that if $F = S \cap A$ with S τ-closed, then F is τ_A-closed. Conversely, suppose F is τ_A-closed, i.e., suppose that its complement $A \backslash F$ in A is τ_A-open. Then $A \backslash F = A \cap U$ with U τ-open. Then $F = A \backslash (A \backslash F) = A \backslash (U \cap A) = A \cap U^c$ (as in the argument with T above), and U^c is τ-closed.

Remarks: If $B \subseteq A \subseteq X$ and τ is a topology on X, then B has a relative topology from τ and a relative topology from τ_A. Fortunately, it is an easy exercise to check that these agree. One can also show easily that if $S \subseteq A \subseteq X$, then the τ_A-closure of S in (A, τ_A) is A intersected with the τ-closure of S in X, and that for each $x \in S$, the τ_A-neighborhoods of x are exactly the intersections of the τ-neighborhoods of x with A. If $\mathcal{B}$ is a base for τ, then $\{B \cap A : B \in \mathcal{B}\}$ is easily seen to be a base for τ_A.

Def: A function $f : (X, \tau) \to (Y, \sigma)$ from one topological space into another is *continuous* if $f(\operatorname{cl} A) \subseteq \operatorname{cl} f(A)$ for all $A \subseteq X$.

Thm. 1.21: A function from (X, τ) into (Y, σ) is continuous if and only if $f^{-1}(U)$ is open in (X, τ) whenever U is open in (Y, σ).

Proof: Observe that for each subset U of Y, $f^{-1}(U)$ is the complement of $f^{-1}(Y \backslash U)$. It follows immediately that the assertion "$f^{-1}(U)$ is open whenever U is open" is equivalent to "$f^{-1}(F)$ is closed whenever F is closed." Thus we need only prove the following result.

Thm. 1.22: A function f from (X, τ) into (Y, σ) is continuous if and only if $f^{-1}(F)$ is closed whenever F is closed.

Proof: Observe that for any set $A \subseteq X$ we have $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\operatorname{cl} f(A))$. Thus if $f^{-1}(F)$ is closed for all closed F, then (as $\operatorname{cl} f(A)$ is closed) $f^{-1}(\operatorname{cl}(f(A))$ is closed for all A. Thus for all A we have $\operatorname{cl} A \subseteq f^{-1}(\operatorname{cl} f(A))$, i.e., $f(\operatorname{cl} A) \subseteq \operatorname{cl}(f(A))$. Conversely, suppose $f(\operatorname{cl} A) \subseteq \operatorname{cl} f(A)$ for all A. Then $f(\operatorname{cl}(f^{-1}(F)) \subseteq \operatorname{cl}(f(f^{-1}(F))) \subseteq \operatorname{cl} F$ (since $f(f^{-1}(F)) \subseteq F$), i.e., $\operatorname{cl}(f^{-1}(F)) \subseteq f^{-1}(\operatorname{cl} F)$. Now F is closed $\Leftrightarrow F = \operatorname{cl} F$, so when F is closed we have $\operatorname{cl}(f^{-1}(F)) \subseteq f^{-1}(F)$. Thus if F is closed, then $f^{-1}(F) = \operatorname{cl}(f^{-1}(F))$, i.e., $f^{-1}(F)$ is closed.

Remark: It is easy to see that f is continuous $\Leftrightarrow$ whenever $x \in X$ and V is a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of x. (Proof: $\Rightarrow$: Let $f(x) \in \mathcal{O} \subseteq V$ with $\mathcal{O}$ open. Then $f^{-1}(\mathcal{O})$ is open and $x \in f^{-1}(\mathcal{O}) \subseteq f^{-1}(V)$. $\Leftarrow$: Let $\mathcal{O}$ be open. For each $x \in f^{-1}(\mathcal{O})$, we have $f(x) \in \mathcal{O}$, so $\mathcal{O}$ is an open neighborhood of $f(x)$. Thus $f^{-1}(\mathcal{O})$ is a neighborhood of x. Since $x \in f^{-1}(\mathcal{O})$ is arbitrary, $f^{-1}(\mathcal{O})$ is a neighborhood of each of its points, hence is open.)

Lemma: Let (X, ρ) be a pseudometric space, $\epsilon > 0$, and $x \in X$. If $y \in C(x, \epsilon)$, then there exists $\delta > 0$ with $C(y; \delta) \subseteq C(x; \epsilon)$.

Proof: Choose δ with $0 < \delta < \epsilon - \rho(x, y)$. Then $\rho(z, y) < \delta$ implies $\rho(z, x) \leq \rho(z, y) + \rho(y, x) < \delta + \epsilon - \delta = \epsilon$. Thus $C(y; \delta) \subseteq C(x; \epsilon)$.

Thm. 1.23: Suppose $f : (X, \rho) \to (Y, d)$ is a function from one pseudometric space into another. Then f is continuous if and only if for each $x \in X$ and each $\epsilon > 0$, there exists $\delta > 0$ so that $d(f(x), f(y)) < \epsilon$ whenever $\rho(x, y) < \epsilon$. (Note δ may depend on x and on ϵ.)

Note: This latter condition says precisely this: for each cell $C(f(x); \epsilon)$ centered at a point $f(x)$ in Y, the inverse image of $C(f(x); \epsilon)$ under f contains a cell $C(x; \delta)$ centered at x (for some choice of $\delta > 0$, δ dependent on x and ϵ).

Proof of Theorem: $\Rightarrow$ Let $x \in X$, $\epsilon > 0$ be given. Then $C(f(x); \epsilon)$ is open, so $f^{-1}(C(f(x); \epsilon))$ is open (and contains x). Thus $f^{-1}(C(f(x); \epsilon))$ is a union of basic sets, i.e., of cells. Thus there exist $\eta > 0$ and $y \in X$ with $x \in C(y; \eta) \subseteq f^{-1}(C(f(x); \epsilon))$. Apply the lemma to get $C(x, \delta) \subseteq f^{-1}(C(f(x); \epsilon))$ as required.

$\Leftarrow$: Let $\mathcal{O}$ be open in (Y, d). Let $x \in f^{-1}(\mathcal{O})$. It suffices to show $f^{-1}(\mathcal{O})$ is a neighborhood of x. Now $x \in f^{-1}(\mathcal{O}) \Rightarrow f(x) \in \mathcal{O}$, so there exists a cell C with $f(x) \in C \subseteq \mathcal{O}$ (by definition of base). By the lemma, there exists $\epsilon > 0$ such that $C(f(x); \epsilon) \subseteq \mathcal{O}$. By assumption, there exists $\delta > 0$ such that f maps $C(x; \delta)$ into $C(f(x); \epsilon)$, i.e., such that $C(x, \delta) \subseteq f^{-1}(C(f(x); \epsilon)) \subseteq f^{-1}(\mathcal{O})$. Thus $f^{-1}(\mathcal{O})$ is a neighborhood of x as required.

Prop. 1.24: Let T be a collection of topologies for a set X. Then $\cap T$ is a topology for X.

Proof: $\phi \in \tau$ and $X \in \tau$ for each τ in T. If $U_1, U_2 \in \tau$ for each τ in T, then so is $U_1 \cap U_2$. If $\mathcal{O} \subseteq \cap T$, then $\mathcal{O} \subseteq \tau$ for each $\tau \in T$, so $\cap\mathcal{O} \in \tau$ for each $\tau \in T$, so $\cap\mathcal{O} \in \cap T$.

Def: Let $\mathcal{C}$ be a collection of subsets of a set X. Then $\tau_{\mathcal{C}} = \cap\{\tau : \mathcal{C} \subseteq \tau,\ \tau \text{ is topology for } X\}$ is called the topology *generated* by $\mathcal{C}$ (or *subgenerated by* $\mathcal{C}$), and $\mathcal{C}$ is said to be a *subbase* for $\tau_{\mathcal{C}}$. (Note that there is always at least one topology τ containing $\mathcal{C}$, namely the power set 2^X.) Clearly $\tau_{\mathcal{C}}$ is the *smallest* topology on X containing $\mathcal{C}$.

Prop. 1.25: The collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{C}$ is a base for the topology generated by $\mathcal{C}$. (That is, $\tau_{\mathcal{C}}$ = all unions of all finite intersections of elements of $\mathcal{C}$.)

Proof: Note first that ϕ is a finite subset of $\mathcal{C}$, and that $\bigcap_{S \in \phi} S = \{x \in X : x \in S \text{ for all } S \in \phi\}$. If now $x \in X$, then for each $S \in \phi$ (there are none!) we have $x \in S$. Thus $\bigcap_{S \in \phi} S \supseteq X$. Clearly $\bigcap_{S \in \phi} S \subseteq X$ (see above), so $X = \bigcap_{S \in \phi} S$ is an intersection of finitely many (namely none) elements of $\mathcal{C}$. Now suppose F_1 and F_2 are finite subsets of $\mathcal{C}$, and let $B_1 = \cap F_1$, $B_2 = \cap F_2$. Then $B_1 \cap B_2 = \bigcap\{S : S \in F_1 \text{ or } S \in F_2\} = \bigcap\{S : S \in F_1 \cup F_2\}$. Since $F_1 \cup F_2$ is a finite set of elements of $\mathcal{C}$, $B_1 \cap B_2$ is in $\mathcal{B}$. In particular if $p \in B_1 \cap B_2$, then there exists $B_3 (= B_1 \cap B_2)$ with $p \in B_3 \subseteq B_1 \cap B_2$ and $B_3 \in \mathcal{B}$. Thus $\mathcal{B}$ is a base for a topology σ on X. Clearly $\mathcal{C} \subseteq \mathcal{B} \subseteq \sigma$ (since each singleton from $\mathcal{C}$ is a finite subset of $\mathcal{C}$, and by definition of the topology generated by a base). It follows that $\tau_{\mathcal{C}} \subseteq \sigma$ (since $\tau_{\mathcal{C}} = \bigcap\{\tau : \tau \supseteq \mathcal{C}$ and τ is a topology$\}$). On the other hand, since each B in $\mathcal{B}$ is a finite intersection of elements of $\mathcal{C}$ and $\mathcal{C} \subseteq \tau_{\mathcal{C}}$, each B in $\mathcal{B}$ lies in $\tau_{\mathcal{C}}$. Thus any union of elements of $\mathcal{B}$ lies in $\tau_{\mathcal{C}}$. Thus $\sigma \subseteq \tau_{\mathcal{C}}$.

Def: Let X be a set and let $\{(Y_\alpha, \tau_\alpha) : \alpha \in A\}$ be a collection of topological spaces. Suppose that for each $\alpha \in A$, there is a function $f_\alpha : X \to Y_\alpha$. The smallest topology (i.e., the intersection of all such) for which every f_α is continuous is the *weak topology on X from (or by) the collection $\mathcal{F} = \{f_\alpha : \alpha \in A\}$*. (Clearly this intersection makes each f_α continuous.)

Note: Every function *out* of the discrete topology is continuous, so there is at least one such topology.

Examples: The product topology on $\mathbb{R}^n$ is the weakest topology making each coordinate projection continuous. The weak topology on a Banach space or Hilbert space is the weakest topology making each element of the dual Hilbert space continuous.

Thm. 1.26: The topology generated by $\{f_\alpha^{-1}(U_\alpha) : \alpha \in A, U_\alpha \in \tau_\alpha\}$ is the weak topology by $\mathcal{F} = \{f_\alpha : \alpha \in A\}$.

Proof: Let τ be the topology generated by $\{f_\alpha^{-1}(U_\alpha) : \alpha \in A, U_\alpha \in \tau_\alpha\}$ and σ be the weak topology by $\mathcal{F}$. Clearly each f_α is a τ-continuous, since $f_\alpha^{-1}(U_\alpha)$ is always τ-open. Thus $\sigma \subseteq \tau$. But τ is precisely the collection of all unions of intersections of finite sets of inverse images $f_\alpha^{-1}(U_\alpha)$, where $\alpha \in A$ and $U_\alpha \in \tau_\alpha$. Each of these inverse images lies in σ (since σ makes each f_α continuous), so unions of finite intersection of these inverse images lie in σ. That is, $\tau \subseteq \sigma$.

Def: A one-to-one continuous function f from one topological space onto another such that its inverse function f^{-1} is also continuous is called a *homeomorphism*, and if such a homeomorphism exists, then the two spaces are said to be *homeomorphic*, or *topologically* *equivalent.* A homeomorphism from one space X onto a subspace of a space Y is called a *homeomorphic embedding*, or a *topological embedding*, or sometimes just an *embedding*, of X into Y.

Example $x \to e^x$ embeds $\mathbb{R}$ into $\mathbb{R}$ (or into $[0, +\infty)$) and is a homeomorphism of $\mathbb{R}$ with $\{x \in \mathbb{R} : x > 0\}$. (Its inverse is "log.")

Def. A topological space X is *disconnected* if there exist non-void sets A and B such that $X = A \cup B$ and $A \cap (\operatorname{cl} B) = (\operatorname{cl} A) \cap B = \phi$. A space that is not disconnected is *connected.*

Thm. 2.1. A space is disconnected if and only if it has a non-void proper closed and open subset.

Note: If a space contains a non-void proper closed and open set, then it must contain at least two such (consider the complement).

Proof of Thm. 2.1: Suppose A is closed and open, and let $B = X \backslash A$. Then B is open and closed. In particular $A \cup B = X$, $A \cap \operatorname{cl} B = A \cap B = (\operatorname{cl} A) \cap B$ and $A \cap B = \phi$. If $A \neq \phi$ and $A \neq X$, then the same is true of B. Conversely, suppose X is disconnected, and let A and B be as in the definition. Then $\phi = A \cap \operatorname{cl} B \supseteq A \cap B$, so $A \cap B = \phi$. Since $A \cup B = X$, A and B are complements. But $A \cap \operatorname{cl} B = \phi$ implies $\operatorname{cl} B \subseteq X \backslash A = B$, so B is closed. Similarly A is closed. Thus A and B are also open. Since $A \neq \phi$ and $B \neq \phi$, $\phi \neq A \neq X$.

Prop. 2.2: The open interval $(0, 1)$ is connected (with the relative topology from the usual topology on $\mathbb{R}$).

Proof: Suppose A and B are non-void proper closed and open subsets of $(0, 1)$ with $(0, 1) = A \cup B$ and $A \cap B = \phi$. Observe that since $(0, 1)$ is open in $\mathbb{R}$, A and B are open in $\mathbb{R}$ (as well as relatively open in $(0, 1)$). Choose $a \in A$ and $b \in B$. We may assume $a < b$. Then $b \in \{t \in B : a < t\} \subseteq (0, 1)$. Let $c = \inf\{t \in B : a < t\}$, so that $a \leq c \leq b$. Observe that if $a < x < c$, then $x \notin B$ (since otherwise $c \leq x < c$, from the fact that c is a lower bound for $\{t \in B : a < t\}$). Thus if $a < x < c$, then $x \in A$. That is, if $a < c$, then $(a, c) \subset A$.

Now either $c \in A$ or $c \in B$. If $c \in A$, then $[a, c] \subseteq A$ (either $a = c$ or $(a, c) \cup \{a\} \cup \{c\} \subset A$). In this case, since A is open, there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq A$, whence $[c, c + \delta) \subseteq [a, c + \delta) \subseteq A$ (since $[a, c + \delta) = [a, c] \cup [c, c + \delta) \subseteq A$). But then $a < t$ and

$t \in B \Rightarrow t \geq c+\delta$ (since $[a, c+\delta) \subseteq A$), so $c+\delta$ is a lower bound for $\{t \in B : t > a\}$, contradicting the fact that c is the greatest such bound. Thus $c \notin A$. Suppose then that $c \in B$. Then since B is open, there exists $\delta > 0$ with $(c-\delta, c+\delta) \subseteq B \subseteq (0,1)$ and we may assume that $a < c-\delta$, whence $a < c - \frac{\delta}{2} < c$. But $c - \frac{\delta}{2} \in B$, so this contradicts an observation above.

Thus $c \notin B$. Thus $c \notin A \cup B$, a clear contradiction.

Remark: It will follow shortly (from a result on closures of connected sets) that $[0,1]$ is connected. We shall also soon see that every homeomorphic (or even continuous) image of a connected set is connected, whence we get that every interval in $\mathbb{R}$, every half-line in $\mathbb{R}$, $\mathbb{R}$ and (of course) ϕ are connected. One can show that these are all the connected subsets of $\mathbb{R}$ (if we agree that a point is an interval). In fact we have:

Exercise: A subset of $\mathbb{R}$ is connected if and only if it is convex.

Prop. 2.3: Let $\Gamma = \{0,1\}$ be a two-point discrete space. (Unique up to homeomorphism!) Then a topological space X is disconnected if and only if there is a continuous function from X onto Γ.

Proof: $\Leftarrow$: If $f : X \to \Gamma$ is onto, then $f^{-1}(0)$ and $f^{-1}(1)$ are complementary non-void $\left\{\begin{matrix}\text{closed}\\ \text{open}\end{matrix}\right\}$ subsets of X (since singletons are $\left\{\begin{matrix}\text{closed}\\ \text{open}\end{matrix}\right\}$ and f is continuous). $\Rightarrow$: Let $X = A \cup B$ with A and B open and closed, $A \cap B = \phi$ and $A \neq \phi, B \neq \phi$. Define $f(x) = 0$ if $x \in A$, $f(x) = 1$ if $x \in B$, so that $f^{-1}(0) = A$, $f^{-1}(B) = 1$. Then $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}\{0,1\} = f^{-1}(0) \cup f^{-1}(1)$ and $f^{-1}(\phi) = \phi$ are all open. Thus, $f^{-1}(\mathcal{O})$ is open for every open $\mathcal{O}$, so f is continuous. Since $A \neq \phi$ and $B \neq \phi$, f is onto.

Thm. 2.7: If A is a connected subset of a space X and $A \subseteq B \subseteq \operatorname{cl} A$, then B is connected.

Lemma: The restriction of a continuous function to a subset A is continuous (out of the relative topology).

Proof: If $\mathcal{O}$ is open in the image space, then $(f|_A)^{-1}(\mathcal{O}) = A \cap f^{-1}(\mathcal{O})$ is relatively open for f continuous.

Lemma: If $A \subseteq B \subseteq X$, then $\operatorname{cl}_B(A) = B \cap \operatorname{cl}_X(A)$.

Pf: $\operatorname{cl}_B(A) = \bigcap\{B \cap K : K \text{ is closed in } X \text{ and } A \subseteq B \cap K\} = \bigcap\{B \cap K : K \text{ is closed in } X \text{ and } A \subseteq K\} = B \cap \{\bigcap\{K : K \text{ is closed in } X \text{ and } A \subseteq K\}\} = B \cap \operatorname{cl}_X(A)$.

Pf. of Thm. 2.7: Let $F : B \to \Gamma = \{0,1\}$ be continuous, where Γ is discrete. Since A

is connected and the restriction of F to A is continuous, F cannot map A onto Γ. Thus $F(A)$ is a singleton, say $\{0\}$. Since F is continuous, $F(\text{cl}_B(A)) \subseteq \text{cl}(F(A)) = \{0\}$. But $\text{cl}_B(A) = (\text{cl}(A)) \cap B = B$, so $F(B) \subseteq \{0\}$. Thus every continuous F from B into Γ fails to be onto (i.e., is constant). That is, B is connected.

Cor: $[0,1]$ (and any interval in $\mathbb{R}$) is connected.

We shall soon see that $\mathbb{R}^n$ is connected for all n.

Thm. 2.8: Let $f : X \to Y$ be continuous, and let X be connected. Then $f(X)$ is connected.

Proof: Let $F = f(X) \to \Gamma = \{0,1\}$ be continuous, where Γ is discrete. Then $F \circ f$ is continuous ($(F \circ f)^{-1}(\mathcal{O}) = f^{-1}(F^{-1}(\mathcal{O}))$ is open for all open $\mathcal{O}$). Since X is connected, $F \circ f$ is constant, so F is constant on $f(X)$. It follows that $f(X)$ is connected.

Remark: We have used the following important (and obvious) fact: composition of two continuous functions yields a continuous function. (Use the open set argument as above.)

Thm. 2.4: Let $\mathcal{C}$ be a collection of connected subsets of a space X, and suppose there exists $\tilde{C} \in \mathcal{C}$ such that $\tilde{C} \cap C \neq \phi$ for every $C \in \mathcal{C}$. Then $\cup\mathcal{C}$ is connected.

Proof: As usual, let $F : \cup\mathcal{C} \to \Gamma = \{1,2\}$, where Γ is discrete and F is continuous. Then for each $C \in \mathcal{C}$, $F(C)$ is connected, hence is a singleton in Γ. Since each $F(C)$ meets $F(\tilde{C})$, they are all the same singleton. Thus F is constant on $\cup\mathcal{C}$, so $\cup\mathcal{C}$ is connected.

Thm. 2.5: Let $\mathcal{C}$ be a collection of connected subsets of a space X, and suppose $C \cap D \neq \phi$ for all $C, D \in \mathcal{C}$. Then $\cup\mathcal{C}$ is connected.

Proof: Modify the proof of Thm. 2.4 in the obvious way.

Cor: If $\mathcal{C}$ is a collection of connected sets and $\cap\mathcal{C} \neq \phi$, then $\cup\mathcal{C}$ is connected.

By using overlapping intervals one now sees from Thm. 2.5 that $\mathbb{R}$ and any set of the form $(a, +\infty)$ or $(-\infty, a)$ are connected. (Hence also $(-\infty, a]$ and $[a, +\infty)$ are connected.) Using overlapping balls shows that $\mathbb{R}^n$ is connected. (Why precisely? Exercise.)

Remark: One can show that $\mathbb{R}$, ϕ, rays in $\mathbb{R}$ and intervals in $\mathbb{R}$ are the *only* connected subsets of $\mathbb{R}$. (Exercise: Show that if C is connected and $a < b$ in C, then $(a,b) \subseteq C$.)

Prop. 2.9: Let X be a topological space and define $R \subseteq X \times X$ by $R = \{(a,b) \in X \times X : \exists$ a connected set in X containing a and $b\}$. Then R is an equivalence relation on X.

Pf: $\{a\}$ is connected, so $(a,a) \in R$ for all $a \in X$. The relation is clearly symmetric.

Transitivity follows from Thm. 2.5.

Def: The equivalence classes of the relation defined in Prop. 2.9 are called *components*, or *connected components*, of X. Observe that unless $X = \phi$, each component of X is non-empty.

If A is connected and A meets a component C of X, then by transitivity of the relation R above, we must have $A \subseteq C$. In particular, if A is connected and contains a component of C of X, then A must coincide with X.

Thm. 2.10: Each component of a topological space X is connected and closed. If $A \subseteq X$ is connected and non-empty, then A is contained in one and only one component of X. A connected subset C of X is a component of X if and only if it has the following property: whenever $C \subseteq A \subseteq X$ and A is connected, then $C = A$. (That is, the components of X are precisely the maximal connected subsets of X.) Finally, X is connected if and only if X has exactly one connected component.

Proof: Let C be a component of X. Let $x \in C$. Then for each $y \in C$, there exists a connected subset K_y of X such that $x \in K_y$ and $y \in K_y$. Clearly K_y meets C, so $K_y \subseteq C$. Thus $\bigcup\{K_y : y \in C\} \subseteq C \subseteq \bigcup\{y : y \in C\} \subseteq \bigcup\{K_y : y \in C\}$, so $C = \bigcup\{K_y : y \in C\}$. Since $\bigcap\{K_y : y \in C\} \supseteq \{x\} \neq \phi$, C is connected (by Thm. 2.4). Thus each component of X is connected.

Since closure of connected sets are connected, and since $C \subseteq \mathrm{cl}(C)$, it follows from a remark preceding the theorem that $C = \mathrm{cl}(C)$ whenever C is a component. Thus components are always closed. It follows from the same remark above that components are maximal connected subsets of X. Each non-empty connected subset of X must be part of an equivalence class under R. Since equivalence classes are either disjoint or coincident, each non-empty connected subset of X is contained in a unique component of X. It follows immediately that a maximal connected subset of X is a component. In particular, X is connected if and only if it has only one connected component.

Def: A continuous $f : [0,1] \to X$ is a *path* in X.

Def: X is *path connected* if for each $x, y \in X$, $\exists$ a path f in X with $f(0) = x$, $f(1) = y$.

Note: If f connects x to y and g connects y to z (in the obvious sense of the last few definitions), then there is a path connecting x to z which can be constructed from f and

g by making use of homeomorphisms from $[0,1/2]$ and $[1/2,1]$ onto $[0,1]$.

Thm. 2.11: Every path connected space is connected.

Pf: Let $a,b \in X$ where X is path connected, and let $f : [0,1] \to X$ with $f(0)=a$, $f(1)=b$, f continuous. Then $f([0,1])$ is connected and contains a and b. Thus X has exactly one component.

Prop. 2.12: Every open connected subset of $\mathbb{R}^n$ is path connected.

Proof: Cells are clearly path connected. Fix $x \in \mathcal{O}$, where $\mathcal{O}$ is open and connected. (We may assume $\mathcal{O} \neq \phi$.) Let $U = \{y \in \mathcal{O} :$ there is a path connecting x and y in $\mathcal{O}\}$. Since $\mathcal{O}$ is open, each $y \in U$ is the center of a cell C contained in $\mathcal{O}$. If $z \in C$, connect x to y in $\mathcal{O}$ and then y to z in $C \subseteq \mathcal{O}$. Thus U is open. Now consider any $z \in \mathcal{O}\backslash U$. Choose a cell C centered at z with $C \subseteq \mathcal{O}$. If $\exists\, y \in U \cap C$, then x and z can be connected (in $\mathcal{O}$) to y, hence to each other. Thus $U \cap C = \phi$. In particular, there exists a neighborhood C of z in $\mathcal{O}$ which does not meet U. Since z is an arbitrary element of $\mathcal{O}\backslash U$, $\mathcal{O}\backslash U$ is open in $\mathcal{O}$, so U is closed in $\mathcal{O}$. Thus U is open and closed in $\mathcal{O}$. Since $\mathcal{O}$ is connected, $U = \mathcal{O}$. (Clearly $x \in U$, so $U \neq \phi$.)

Thm. 2.13: If $f : X \to f(X) = Y$ is continuous and X is path connected, then $f(x)$ is path connected.

Pf: If $a,b \in Y$, choose $x,y \in X$ with $f(x)=a$, $f(y)=b$. Choose a path $F : [0,1]$ in X from x to y. Then $f \circ F$ connects a to b.

Def: Let (X,τ) be a topological space. A subcollection $\mathcal{C}$ of τ is called an *open cover* of X if $X = \cup\mathcal{C}$. A topological space X is *compact* if every open cover $\mathcal{C}$ has a finite subset $\mathcal{F} \subseteq \mathcal{C}$ such that $X = \cup\mathcal{F}$ (i.e., has a finite *subcover*).

Prop. 2.14: A subset A of a topological (X,τ) is compact if and only if for every $\mathcal{C} \subseteq \tau$ such that $A \subseteq \cup\mathcal{C}$, there is a finite $\mathcal{F} \subseteq \mathcal{C}$ such that $A \subseteq \cup\mathcal{F}$. (That is, a subspace is compact in the *relative* topology from τ if and only if it is satisfies this condition on $\mathcal{C}$.)

Proof: Let A be compact, and let $\mathcal{C} \subseteq \tau$ with $A \subseteq \cup\mathcal{C}$. Then $A \subseteq A \cap (\cup\mathcal{C}) = \bigcup\{A \cap C : C \in \mathcal{C}\}$ and each $A \cap C$ is relatively open. Since A is compact, there is a finite subset of $\mathcal{F}$ of $\mathcal{C}$ such that $A \subseteq \bigcup\{A \cap C : C \in \mathcal{F}\} = A \cap \{\cup C : C \in \mathcal{F}\}$, i.e., such that $A \subseteq \{\cup C : C \in \mathcal{F}\} = \cup\mathcal{F}$. Conversely, suppose that whenever $\mathcal{C} \subseteq \tau$ with $A \subseteq \cup\mathcal{C}$, there is a finite $\mathcal{F} \subseteq \mathcal{C}$ with $A \subseteq \cup\mathcal{F}$. Let $\{A \cap C : C \in \mathcal{C}\}$ be any relatively open cover of A with

$A = \bigcup\{A \cap C : C \in \mathcal{C}\}$. Then $A \subseteq \bigcup\{C : C \in \mathcal{C}\} = \cup\mathcal{C}$, so there exists a finite $\mathcal{F} \subseteq \mathcal{C}$ with $A \subseteq \cup\mathcal{F}$. But then $A \subseteq A \cap (\cup\mathcal{F}) = \bigcup\{A \cap C : C \in \mathcal{F}\} \subseteq A$, so $A = \bigcup\{A \cap C : C \in \mathcal{F}\}$.

Thm. 2.15: Every closed subset of a compact space is compact.

Proof: Let $K \subseteq X$ be closed. Let $K \subseteq \cup\mathcal{C}$ where $\mathcal{C} \subseteq \tau$. Let $\mathcal{C}' = \mathcal{C} \cup \{X \backslash K\}$. Then $\mathcal{C}' \subseteq \tau$ since K is closed. Thus there exists a finite $\mathcal{F}' \subseteq \mathcal{C}'$ with $X \subseteq \cup\mathcal{F}'$. Clearly $K \subseteq \cup\mathcal{F}'$, and since $K \cap (X \backslash K) = \phi$, $K \subseteq \cup\mathcal{F}$, where $\mathcal{F} = \mathcal{F}' \backslash \{X \backslash K\}$. Clearly $\mathcal{F}$ is finite.

Def.: A topological space (X, τ) is *Hausdorff* if for every pair x, y in X with $x \neq y$, there are open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \phi$.

Thm. 2.16: Each compact subset of a Hausdorff space (X, τ) is closed.

Proof: Let K be compact and $\mathcal{O} = X \backslash K$. We shall show $\mathcal{O}$ is open. Fix $y \in \mathcal{O}$. For each $x \in K$, open sets U_x and V_x with $x \in U_x$, $y \in V_x$ and $U_x \cap V_x = \phi$. Then $\{U_x : x \in K\}$ is a family of open sets with $K \subseteq \bigcup\limits_{x \in K} K_x$. Thus $\exists\ x_1, \ldots, x_n$ with $K \subseteq \bigcup\limits_{i=1}^{n} U_{x_i} = U$. Let $V = \bigcap\limits_{i=1}^{n} V_{x_i}$. Then $V_x \cap U_x = \phi$ for all $x \in K$ implies $U \cap V = \phi$ (by distributivity of $\cap$ and $\cup$ over one another). Thus $K \cap V = \phi$, so y is an interior point of $\mathcal{O}$. Thus $\mathcal{O}$ is open.

Prop. 2.17: The interval $[0, 1]$ is compact.

Proof: Let $\mathcal{C}$ be a family of open sets with $[0, 1] \subseteq \cup\mathcal{C}$. Let $T = \{x \in [0, 1] : \{t : 0 \leq t \leq x\}$ can be covered by finitely many sets in $\mathcal{C}\}$. Thus $O \in T$, so T is non-empty. Let $c = \sup T$, so that $0 \leq c \leq 1$. We need only show that $c = 1$. Choose $C \in \mathcal{C}$ with $c \in C$. Since C is open, $\exists\ \delta > 0$ with $(c - \delta, c + \delta) \subseteq C$. Since c is the least upper bound for T, $\exists\ x \in (c - \delta, c)$ with $x \in T$. Thus finitely many elements of $\mathcal{C}$ cover $[0, x]$, say a finite subcollection $\mathcal{F}'$. Put $\mathcal{F} = \mathcal{F}' \cup \{C\}$ (so that $\mathcal{F}$ is finite). Then $\mathcal{F}$ covers $[0, c + \delta/2]$. If $c < 1$, this contradicts the fact that c is an upper bound for T. Thus $c = 1$. But then we see that $\mathcal{F}$ covers $[0, 1] = [0, c]$.

Thm. 2.18: Suppose $f : X \to Y$ is continuous and (X, ϵ) is compact. Then $f(X)$ is compact.

Proof: Let $f(X) \subseteq \cup\mathcal{C}$, i.e., $X \subseteq f^{-1}(\cup\mathcal{C}) = f^{-1}\{\cup C : C \in \mathcal{C}\} = \bigcup\{f^{-1}(C) : C \in \mathcal{C}\}$. If $\mathcal{C} \subseteq \tau$, then there exists a finite $\mathcal{F} \subseteq \mathcal{C}$ with $X \subseteq \bigcup\{f^{-1}(C) : C \in \mathcal{F}\} = f^{-1}\{\cup C : C \in \mathcal{F}\} = f^{-1}(\cup\mathcal{F})$. That is, $f(X) \subseteq \cup\mathcal{F}$. Thus $f(X)$ is compact.

Prop. 2.19: Let $f : X \to Y$ be continuous, and suppose X is compact and Y is Hausdorff. Then F closed implies that $f(F)$ is closed.

Proof: F closed in a compact $X \Rightarrow F$ is compact $\Rightarrow f(F)$ is compact (by continuity). But Y is Hausdorff, so $f(F) \subseteq Y$ implies $f(F)$ is closed in Y.

Thm. 2.20: Suppose $f : X \to Y$ is one-to-one and onto, and continuous. If X is compact and Y is Hausdorff, then f is a homeomorphism.

Proof: f^{-1} is continuous $\Leftrightarrow f(F)$ is closed whenever F is closed (since $f = (f^{-1})^{-1}$).

Cor: If τ_1 and τ_2 are topologies on X such that τ_1 is compact and τ_2 is Hausdorff, and if $\tau_1 \supseteq \tau_2$, then $\tau_1 = \tau_2$.

Proof: Let $i(x) = x$ for all x in X. Then i is continuous from (X, τ_1) to (X, τ_2) (since $\tau_1 \supseteq \tau_2$). Clearly i is 1-1 onto. Thus i is a homeomorphism.

Note: If $\tau_1 \subseteq \tau_2$, where τ_1 and τ_2 are topologies on X, then τ_2 compact $\Rightarrow \tau_1$ compact, and τ_1 Hausdorff $\Rightarrow \tau_2$ Hausdorff (Exercise).

Def: A *sequence* in a set S is a function from the positive integers $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ into S. If $f : \mathbb{Z}^+ \to S$ is a sequence, we generally write x_n for $f(n)$ and denote this sequence by $\{x_n\}$ or (x_n) (or $\{x_n\}_{n=1}^{+\infty}$, etc.)

Note: Sometimes domains other than $\mathbb{Z}^+$ are allowed (particularly $\mathbb{Z}$ or $\{0, 1, 2, 3, \ldots\}$.

Def: Let $A \subseteq X$ and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is *eventually* in A if there is an integer N such that $x_n \in A$ for all $n \geq N$. We say that $\{x_n\}$ is *frequently* in A if for each integer $k \geq 1$ there is an integer $n \geq k$ with $x_n \in A$.

Note: For any $A \subseteq X$ and any sequence $\{x_n\}$ in A, either $\{x_n\}$ is eventually in A or $\{x_n\}$ is frequently in $X \backslash A$, but *not* both.

Def: A sequence $\{x_n\}$ in a topological space X is said to *converge* to a point x of X if $\{x_n\}$ is eventually in each neighborhood of x. The sequence is said to *cluster* at x if it is frequently in each neighborhood of x. (Consider $x_n = \frac{1}{n}$ vs. $x_n = (-1)^n$.)

Prop. 3.1: A sequence in a Hausdorff space converge to at most one point.

Proof: Suppose $\{x_n\}$ converges to x and to y. Let U, V be any open neighborhoods of x and y respectively. Then $\exists\ N_1$ and N_2 such that $n \geq N_1 \Rightarrow x_n \in U$, $n \geq N_2 \Rightarrow x_n \in V$. Then $n \geq \max\{N_1, N_2\} \to x_n \in U \cap V$. Thus if x is Hausdorff, then $x = y$.

Def: A sequence $\{z_n\}$ is a *subsequence* of a sequence $\{x_n\}$ if there exists a strictly increasing sequence $\{k_n\}$ of positive integers such that $z_n = x_{k_n}$.

Note: For all n, we must have $k_n \geq n$.

Prop. 3.2: Suppose $\{x_n\}$ is a sequence in a topological space X, and suppose $\{x_{k_n}\}$ is a subsequence of $\{x_n\}$ which converges to a point p of X. Then $\{x_n\}$ clusters at p.

Proof: Let U be a neighborhood of p. Let $m \in \mathbb{Z}^+$. Since $\{x_{k_n}\}$ converges to p, there exists $N \in \mathbb{Z}^+$ such that $n \geq N$ implies $x_{k_n} \in U$. We may assume that $N \geq m$. Then $k_N \geq k_m \geq m$ (since $n \to k_n$ is increasing), and $x_{k_N} \in U$.

Note: If $\{x_n\}$ converges to p, then so does every subsequence of $\{x_n\}$ (Exercise).

Prop. 3.3: Suppose $A \subseteq X$, where X is a topological space, and suppose $\{a_n\}$ is a sequence in A which converges to a point p of X. Then $p \in \operatorname{cl} A$.

Proof: Let U be any neighborhood of p. Then $\exists\ N \in \mathbb{Z}^+$ such that $n \geq N$ implies $x_n \in U$. Thus (since $x_n \in A$) each neighborhood U of p meets A.

Prop. 3.4: Suppose (X, d) is a pseudometric space and $A \subseteq X$. If $p \in \operatorname{cl} A$, then there exists a sequence in A which converges to p.

Proof: Let $n \in \mathbb{Z}^+$, and $C_n = \{x \in X : d(x,p) < \frac{1}{n}\}$. Then C_n is an open neighborhood of p, and so contains a point x_n of A (since $p \in \operatorname{cl} A$). This defines a sequence $\{x_n\}$ in A. If U is any neighborhood of p, then there exists $\epsilon > 0$ such that $C(p;\epsilon) \subseteq U$. As soon as $N \geq \frac{1}{\epsilon}$ (so that $\frac{1}{N} \leq \epsilon$) we have for all $n \geq N$ that $x_n \in C_n \subseteq U$. Thus (x_n) converges to p.

Prop. 3.5: Let $f : x \to y$ be continuous from one topological space to another. Let $\{x_n\}$ be a sequence in X which converge to a point x of X. Then the sequence $\{f(x_n)\}$ in Y converge to $f(x)$.

Proof: Let U be an open neighborhood of $f(x)$. Then $f^{-1}(U)$ is an open neighborhood of x, by continuity of f. Since $\{x_n\}$ converges to x, there exists N such that $n \geq N$ implies $x_n \in f^{-1}(U)$, whence $n \geq N$ implies $f(x_n) \in U$.

Prop. 3.6: If $f : (X, d) \to Y$ is a function from a pseudometric space into a topological space such that for every sequence $\{x_n\}$ that converges to a point x in X, the sequence $\{f(x_n)\}$ converges in Y to $f(x)$. Then f is continuous.

Proof: Let $A \subseteq X$, and consider $f(\operatorname{cl} A)$. If $x \in \operatorname{cl} A$, then there exists a sequence $\{x_n\}$ in A such that $\{x_n\}$ converges to x (by Prop. 3.4). By hypothesis, $\{f(x_n)\}$ converges to $f(x)$.

Let U be any neighborhood of $f(x)$. Then there exists $N \in \mathbb{Z}^+$ so that $n \geq N \Rightarrow f(x_n) \in U$. In particular, each neighborhood U of $f(x)$ meets $f(A)$. Thus $f(x) \subseteq \mathrm{cl}(f(A))$. But this means $f(\mathrm{cl}\, A) \subseteq \mathrm{cl}(A)$ (since $x \in \mathrm{cl}\, A$ is arbitrary).

Def: A sequence $\{x_n\}$ in a pseudometric space (X, d) is a *Cauchy sequence* if for every $\epsilon > 0$ there is an integer $N \in \mathbb{Z}^+$ such that $d(x_n, x_m) < \epsilon$ whenever $n \geq N$ and $m \geq N$.

Note: Every convergent sequence is Cauchy, but the converse can fail (exercises).

Exercise: Every Cauchy sequence that clusters converges.

Thm. 3.7: Every sequence in a compact space clusters.

Proof: Let $\{x_n\}$ be a sequence in X. For each $x \in X$, if $\{x_n\}$ does *not* cluster at x, then there exists an open neighborhood $\mathcal{O}_x$ of x such that $\{x_n\}$ is *not* frequently in $\mathcal{O}_n$, i.e., such that x_n is eventually in $X \backslash \mathcal{O}_x$. Thus if $\{x_n\}$ does not cluster at x, then there exist an open neighborhood $\mathcal{O}_x$ of x and $N_x \in \mathbb{Z}$ such that $n \geq N_x$ implies $x_n \notin \mathcal{O}_x$. Let $\mathcal{F}$ be many finite set of such x, and let $N = \max\{N_x : x \in \mathcal{F}\}$. Then $n \geq N_x$ implies $x_n \notin \bigcup_{x \in \mathcal{F}} \mathcal{O}_x$. Thus, $\{x_n\}$ has *no* cluster points in X, then $\{\mathcal{O}_x : x \in X\}$ is an open cover for X and x_n lie eventually outside the union $\bigcup_{x \in \mathcal{F}} \mathcal{O}_x$ for any finite $\mathcal{F}$. It follows then that unless $\{x_n\}$ clusters somewhere, X has an open cover with no finite subcover.

Def: A pseudometric space is *complete* if every Cauchy sequence in it converges.

Prop. 3.8: Every compact pseudometric space is complete.

Proof: Every Cauchy sequence in a compact space clusters, and every Cauchy sequence which clusters converges.

Def: Let S be a nonempty subset of a pseudometric space (X, d). Then the *diameter* of S is $\mathrm{diam}\, S = \sup\{d(x, y) : x, y \in S\}$. (This may be $+\infty$.)

Thm. 3.9: Let (X, d) be a complete metric space, and let $f : X \to X$ be continuous. Suppose there exists a real number α with $0 \leq \alpha < 1$ such that $d(f(x), f(y)) \leq ad(x, y)$ for all x, y in X. Then there exists exactly one $z \in X$ such that $f(z) = z$. Moreover, if $x \in X$, then the sequence $\{f^n(x)\}$ converges to z. (This is the Banach Fixed Point Theorem.)

Proof: By induction, we get easily that for all $k \geq 1$, $d(f^k(x), f^k(y)) \leq \alpha^k d(x, y)$. Thus also if $1 \leq m < n$ in $\mathbb{Z}^+$, we have $d(f^m(x), f^n(x)) \leq \sum_{j=0}^{n-1-m} d(f^{m+j}(x), f^{m+j+1}(x)) \leq (\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}) d(x, f(x)) = \alpha^m [1 + \cdots + \alpha^{n-m-1}] d(x, f(x)) < \frac{\alpha^m}{1-\alpha} d(x, f(x)) \to 0$. Thus $\{f^m(x)\}$ is Cauchy, so converge to some z, by completeness. Now f is continuous, so $f(f^m(x))$ converges to $f(z)$, i.e., $\{f^{m+1}(x)\}$ converges to $f(z)$. But $\{f^{m+1}(x)\}$ is a subsequence of $\{f^m(x)\}$, so has the same limit. Thus, since metric spaces are Hausdorff, $z = f(z)$. If also $f(w) = w$, then $f^k(w) = w$ and $f^k(z) = z$ for all k, whence $d(z, w) = d(f^k(z), f^k(w)) \leq \alpha^k d(z, w)$. As $\alpha^k \to 0$, we must have $d(z, w) = 0$, so $z = \alpha$.

Prop. 3.10: Let (X, d) be a complete pseudometric space, and suppose $\{S_n : n \in \mathbb{Z}_+$ is a collection of non-empty closed subsets of X such that $S_n \supseteq S_{n+1}$ for all $n \geq 1\}$. Suppose also that lim diam $S_n = 0$. Then $\bigcap_n S_n \neq \phi$.

Proof: Choose $x_n \in S_n$ for all n. Then $d(x_n, x_m) \leq \operatorname{diam} S_n$ whenever $n \leq m$. Since diam $S_n \to 0$, $\{x_n\}$ is Cauchy, so converges by completeness to some $x \in S$. Now $n \leq m \Rightarrow x_m \in S_n$, and S_n is closed, so $x \in S_n$ for all n. (Note that $\{x_m : m \geq n\}$ is a sequence in S_n: renumber the indices.)

Def: A subset S of a topological space X is *dense in* X if $\operatorname{cl} S = X$.

Exercise: S is dense in $X \Leftrightarrow$ every non-empty open subset of X meets S.

Thm. 3.11: Let (X, d) be a complete pseudometric space, and let $\{D_n = n \in \mathbb{Z}^+\}$ be a (countable) collection of open dense subsets of X. Then $\cap D_n$ is a dense subset of X.

Proof of Thm. 3.11: Let $B(x; r) = \{y \in X : d(x, y) \leq r\}$, $C(x; r) = \{y \in X : d(x, y) < r\}$. Let $U \neq \phi$ be open. We need to show U meets $\bigcap_n D_n$. Since D_1 is dense, $D_1 \cap U \neq \phi$. Let $x_1 \in D_1 \cap U$. Choose r_1 with $0 < r_1 < 1$ so that $B(x_1; r_1) \subseteq U \cap D_1$ ($U \cap D_1$ is open, so we can choose $r_1 > 0$ with $C(x_1; r_1) \subseteq U \cap D_1$. Choosing a slightly smaller r_1 will ensure $B(x_1; r_1) \subseteq U \cap D_1$). Now $C(x_1; r_1) \subseteq B(x_1; r_1)$ and $C(x_1, r_1)$ is open. Since D_2 is dense, $C(x_1, r_1) \cap D_2$ is non-empty. It is also open. Thus $\exists\ r_2$ and $x_2 \in C(x_1; r_1) \cap D_2$ with $0 < r_2 < \frac{1}{2}$, $C(x_2; r_2) \subseteq B(x_2; r_2) \subseteq C(x_1; r_1) \cap D_2$. Continuing inductively, we get a sequence $B(x_n; r_n)$ such that $0 < r_n < \frac{1}{n}$ and $C(x_{n+1}; r_{n+1}) \subseteq B(x_{n+1}; r_{n+1}) \subseteq C(x_n; r_n) \cap D_{n+1}$ for all n; as before each $C(x_n; r_n) \cap D_{n+1}$ is open and non-empty. Note that $B(x_{n+1}; r_{n+1}) \subseteq B(x_n; r_n) \subseteq U \cap D_1 \subseteq U$ for all $n \geq 1$. We claim $\{x_n\}$ is Cauchy. If $\epsilon > 0$, choose N so that $\frac{2}{N} < \epsilon$. If then $n > N$ and $m > N$, then also x_n and x_m lie in $B(x_N, r_N)$ and so $d(x_n, x_m) \leq d(x_n, x_N) + d(x_N, x_m) < r_N + r_N < \frac{1}{N} + \frac{1}{N} < \epsilon$. Thus $\{x_n\}$ is Cauchy.

Thus $\{x_n\}$ converges to some z in X. We claim $z \in \bigcap_n B(x_n; r_n)$. Indeed, let $N \geq 1$. Then $x_n \in B(x_n; r_n) \subseteq B(x_N; r_N)$ for all $n \geq N$. The sequence $\{x_{N+k} : k \geq 1\}$ is a subsequence of $\{x_n\}$, so converges to z. It is also a sequence in the closed set $B(x_N; r_N)$, so that $z \in B(x_N; r_N)$. Since $N \geq$ is arbitrary, $z \in \bigcap_n B(x_n, r_n)$. Thus $z \in \bigcap_n D_n$ (since $B(x_n, r_n) \subseteq D_n$ for all n). But $B(x_n; r_n) \subseteq U$ for all $U \Rightarrow z \in U$. Then U meets $\bigcap_n D_n$ as required.

Thm. 3.12: Let (X, d) be a complete pseudometric space. Suppose $\{F_n : n \in \mathbb{Z}^+\}$ is a (countable) collection of closed subsets of X, each with empty interior. Then $\bigcup_n F_n$ has empty interior.

Proof: Exercise. Begin by showing A is open dense $\Leftrightarrow X \backslash A$ is closed with empty interior.

Remark: Theorems 3.11 and 3.12 are forms of the Baire Category Theorem.

Def: A subset E of X is *nowhere dense* if its closure contains no nonvoid open set, i.e., if the complement of its closure is dense. A subset E of X is *meager* (or *of first category*) if it is the union of a countable collection of nowhere dense sets. The complement of a meager set is called *residual*. We then have the following version of the *Baire category theorem*:

Cor: Let X be a complete pseudometric space. Then no non-empty open subset of X is meager. That is, no non-empty open subset of X is the union of a countable collection of nowhere dense (or even meager) subsets.

Pf: Let $\{E_n\}$ be a countable collection of nowhere dense subsets, and let U be a non-empty open set. Then $\mathcal{O}_n = X \backslash \operatorname{cl}(E_n)$ is open and dense in X. Thus $\bigcap_n \mathcal{O}_n$ meets U. But if $x \in U \left(\bigcap_n \mathcal{O}_n \right)$, then $x \notin UE_n$, so U is not contained in UE_n.

Cor: Let $\mathcal{F}$ be a family of real-valued continuous functions on a complete pseudometric space, and suppose that for each $x \in X$ $\exists$ M_x such that $|f(x)| \leq M_x$ for all $f \in \mathcal{F}$. Then $\exists$ a non-empty open set $\mathcal{O} \subseteq X$ and a constant M such that $|f(x)| \leq M$ for all $f \in \mathcal{F}$ and all $x \in \mathcal{O}$.

Pf: Exercise.

Def: Let $\{X_\alpha : \alpha \in \Lambda\}$ be a collection of sets. The set Z of all functions f from Λ into $\bigcup\{X_\alpha : \alpha \in \Lambda\}$ such that $f(\alpha) \in X_\alpha$ for all $\alpha \in \Lambda$ is called the *product* (or *Cartesian product*) of the collection. We write $Z = \prod\{X_\alpha : \alpha \in \Lambda\}$ or $Z = \prod\limits_{\alpha\in\Lambda} X_\alpha$ (or sometimes $Z = \mathop{\times}\limits_{\alpha\in\Lambda} X_\alpha$, but usually not with X's). For each $\alpha_0 \in \Lambda$, the function $\pi_{\alpha_0} : \prod\{X_\alpha\} \to X_{\alpha_0}$ given by $\pi_{\alpha_0}(f) = f(\alpha_0)$ is the *projection* of $\prod\{X_\alpha\}$ onto X_{α_0}.

Remark: If Λ is infinite, we need some assurance that there exist functions in $\prod\{X_\alpha\}$. We generally assume the existence by invoking the Axiom of Choice, which says that if $\Lambda \neq \phi$ and each X_α is non-void, then the product $\prod\{X_\alpha\}$ is non-void. More colorfully (but less precisely) this axiom says that for any non-empty collection of non-empty sets there exists a function (a "choice function") which chooses exactly one element from each factor X_α. (Any f in the product is such a choice function.)

Def: If each X_α is a topological space, the weak topology for $\prod\{X_\alpha\}$ by the collection $\{\prod_\alpha : \alpha \in \Lambda\}$ is called the *product topology.*

Prop. 4.1: Let $Z = \prod\{X_\alpha : \alpha \in \Lambda\}$ and $\beta \in \Lambda$. Let $U_\beta \subseteq X_\beta$. Then $\pi_\beta^{-1}(U_\beta) = \prod\{Y_\alpha : \alpha \in \Lambda\}$, where $Y_\beta = U_\beta$ and $Y_\alpha = X_\alpha$ for all $\alpha \neq \beta$.

Proof: Exercise. (This is almost obvious: just use the definitions).

Prop. 4.2: The collection of all sets $\prod\{U_\alpha : \alpha \in \Lambda\}$, where U_α is an open subset of X_α and $U_\alpha = X_\alpha$ for all but a finite number of indices α, is a base for the product topology on $\prod\{X_\alpha : \alpha \in \Lambda\}$. [Note: An arbitrary product of open sets need *not* be open.]

Proof: The sets in this collection are precisely all finite interactions of sets of the form $\pi_\beta^{-1}(U_\beta)$, where U_β is open in X_β, by the last proposition. But this is precisely a base for the weak topology by the functions $\{\pi_\alpha : \alpha \in \Lambda\}$.

Note: If Λ is finite, this base consists of all "open rectangles" (or "boxes", etc.)

Thm. 4.3: Suppose X is endowed with the weak topology by a collection $\{f_\alpha : \alpha \in \Lambda\}$ of functions where $f_\alpha : X \to Y_\alpha$ for all α. If $f : Z \to X$ is a function from a topological space Z into X, then f is continuous if and only if each composition $f_\alpha \circ f : Z \to Y_\alpha$ is continuous.

Proof: Exercise: show that f is continuous $\Leftrightarrow f^{-1}(C)$ is open for each subbasic set C.

Cor: 4.4: If $f : Z \to \prod\{X_\alpha : \alpha \in \Lambda\}$ is a function from a topological space Z into a product space, then f is continuous if and only if each "coordinate" function $\pi_\alpha \circ f$ is continuous.

Prop. 4.5: A countable product of pseudometric spaces is a pseudometric space.

Proof: Let $(X_i, d_i)_{i=1}^{+\infty}$ be pseudometric spaces. We must find a pseudometric for the product topology (i.e., which induces the product topology) on $Z = \prod\limits_i (X_i, d_i)$. Observe first that if (X, d) is a pseudometric space, then $d'(x, y) = \min(d(x, y), 1)$ is a pseudometric on X with $d' \leq d$. In particular, $d(x_n, x) \to 0 \Rightarrow d'(x_n, x) \to 0$. But if $d'(x_n, x) \to 0$, then eventually $d'(x_n, x) = d(x_n, x)$ and so $d(x_n, x) \to 0$. Thus d' and d are equivalent pseudometrics. (By Proposition 3.3 and 3.4, they induce topologies which have the same closed sets.) Thus we may assume that for each i, $d_i \leq 1$. Now define $d(x, y) = \sum\limits_{i=1}^{+\infty} \frac{1}{2^i} d_i(x(i), y(i))$, where $x, y \in Z$. If now $\epsilon > 0$, and if $d(x, y) < \epsilon/2^i$, then $d_i(x(i), y(i)) < \epsilon$. Thus the topology induced by d makes each π_i continuous, so contains the product topology (i.e., the weak topology by $\{\pi_i : i \in \mathbb{Z}^+\}$). Conversely, let $C = C(x; r)$ be a cell in (Z, d), and we shall show that C is open in the product topology (whence it follows that the topology induced by d consists of sets which are open in the product topology, and we will be done). Let $y \in C(x; r)$ and choose $\epsilon > 0$ so that $C(y; \epsilon) \subseteq C(x; r)$. Let N be such that $2^{-N} = I_{n=N+1}^{+\infty} \frac{1}{2^n} < \frac{\epsilon}{2}$. For each $n = 1, 2, 3, \ldots, N$, let

$$U_n = \left\{ z \in X_n : \frac{d_n(y(n), z)}{2^n} < \frac{\epsilon}{2N} \right\}.$$

Thus $B = \bigcap\limits_{n=1}^{N} \pi_n^{-1}(U_n)$ is open in the product topology and contains y. It suffices to show that $B \subseteq C(x; r)$. Let $p \in B$. Then $\pi_n(p) \in U_n$ for $n = 1, 2, \ldots, N$, so

$$\frac{d_n(\pi_n(y), \pi_n(p))}{2^n} = \frac{d_n(y(n), p(n))}{2^n} < \frac{\epsilon}{2N}.$$

Now

$$\begin{aligned} d(y, p) = \sum_{n=1}^{+\infty} \frac{d_n(y(n), p(n))}{2^n} &= \sum_{n=1}^{N} \frac{d_n(y(n), p(n))}{2^n} + \sum_{n=N+1}^{+\infty} \frac{d_n(y(n), p(n))}{2^n} \\ &\leq \sum_{n=1}^{N} \frac{\epsilon}{2N} + \sum_{n=N+1}^{+\infty} \frac{1}{2^n} = \frac{\epsilon}{2} + \frac{1}{2^N} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

(since $d_n \leq 1$ for all n).

Note: It is easy to check that if each d_i is a metric, then so is the d defined above. Thus a countable product of metric spaces is a metric space. It is also easy to modify the argument above to show that if Z is a finite product of pseudometric (or metric) spaces, say $(X_i, d_i)_{i=1}^N$, then we get a pseudometric (or metric) for Z by defining

$$d(x, y) = \sum_{i=1}^{N} d_i(x(i), y(i)) \qquad \text{(exercise)}.$$

In particular the product topology on $\mathbb{R}^n$ is induced by the metric ρ given by

$$\rho(x, y) = \sum_{i=1}^{n} |x(i) - y(i)|$$

Prop. 4.6: Any product of Hausdorff spaces is Hausdorff.

Proof: Exercise.

Prop. 4.7: The usual metric of $\mathbb{R}^n$ induces the product topology on $\mathbb{R}^n$.

Proof of 4.7: As noted above, $\rho(x, y) = \sum_{i=1}^n |x(i) - y(i)|$ induces the product topology on $\mathbb{R}^n$. Now $\rho(x, y) \leq n \max\limits_{1 \leq i \leq n} |x(i) - y(i)| \leq n d(x, y)$, where $d(x, y) = \left(\sum_{i=1}^n |x(i) - y(i)|^2\right)^{1/2}$, since extracting square roots is monotone. But $(d(x, y))^2 = \sum\limits_{1}^{n} |x(i) - y(y)|^2 \leq n \max |x(i) - y(i)|^2$ so $d(x, y) \leq \sqrt{n} \max |x(i) - y(i)| \leq \sqrt{n} \sum_{i=1}^n |x(i) - y(i)|$. In particular there are constants $k > 0$ and $c > 0$ such that $\rho(x, y) \leq d(x, y)$ and $d(x, y) \leq k\rho(x, y)$ for all $x, y \in \mathbb{R}^n$. It follows that ρ and d induces the same topology. (They make the same sequences converge to the same limits, or observe that each point in a cell for one metric is the center of a cell for the other metric.)

Thm. 4.8: (Tychnoff's Theorem): The product of any collection of compact spaces is compact.

Pf: See below.

Thm. 4.9: A subset of $\mathbb{R}^n$ is compact $\Leftrightarrow$ it is closed and bounded (w.r.t. the usual metric). (This is the Heine-Borel Theorem.)

Thm. 4.10: Let $\mathbb{R}^n$ have the usual metric. Then every Cauchy sequence in $\mathbb{R}^n$ is bounded.

Thm. 4.11: Let $\mathbb{R}^n$ have the usual metric. Then $\mathbb{R}^n$ is complete.

Proofs of 4.9, 4.10, 4.11: Exercises.

To prove Tychnoff's Theorem we will employ Zorn's Lemma, which is a variant of the Axiom of Choice. (They are equivalent, in fact: see Halmos' *Naive Set Theory* or Hewitt and Stromberg, *Real and Abstract Analysis.*)

Def: Let X be a set. A relation R in X is a *partial ordering* of X if R is reflexive (xRx for all $x \in X$), antisymmetric (xRy and yRx imply $x = y$), and transitive (xRy and yRz imply xRz). A *partially ordered set* is a pair (X, R) consisting of a set X and a partial ordering of X. A partial ordering R on X is a *total order* (or *linear ordering*) if for each pair x and y in X either xRy or yRx. An element x of a partially ordered set (X, R) is an *upper bound* for a subset S of X if sRx for all $s \in S$. An element x of a partially ordered set (X, R) is a *maximal element* if xRy and $y \in X$ imply that $x = y$.

Note: If R is a partial ordering on X, then the relation T defined by $xTy \Leftrightarrow yRx$ is also a partial ordering on X, and R is a total order $\Leftrightarrow$ T is a total order. A *lower bound for* S with respect to R is an upper bound for S with respect to T. A *minimal element* for X with respect to R is a maximal element of X with respect to T. Observe that passage from R to T turns upper bounds to lower bounds, maximal elements to minimal elements, and vice versa.

Note: A maximal element in a partially ordered set need not be unique (but two maximal elements must coincide whenever they are related).

Zorn's Lemma: Let (X, R) be a non-void partially ordered set. Suppose that each linearly ordered subset $\mathcal{L}$ of X has an upper bound in X. Then X has a maximal element.

The following proof of the Tychnoff Theorem is taken from Hewitt and Stromberg, *Real and Abstract Analysis.* Its core is the following lemmas.

Lemma (Alexander): Let (X, τ) be a topological space, and let $\mathcal{S}$ be any subset of τ such that τ is generated by $\mathcal{S}$ (i.e., such that each element of τ is a union of sets each of which is a finite intersection of elements of $\mathcal{S}$). Then the following are equivalent

i) The space X is compact.

ii) Every cover of X by a subfamily of $\mathcal{S}$ has a finite subcover.

Proof: Clearly i) implies ii), so assume ii) holds. Let us suppose i) fails. Then the family $\mathcal{Z}$ of all open covers of X which do not have a finite subcover is not empty. This family $\mathcal{Z}$ is partially ordered by set theoretic inclusion. Clearly the union of a non-void linearly ordered subset $\mathcal{L}$ of $\mathcal{Z}$ has an upper bound in $\mathcal{Z}$ (namely the union of all the elements of $\mathcal{L}$: any finite subset of this union lies in one element of $\mathcal{L}$ because of the total ordering). Then $\mathcal{Z}$ contains a maximal element, i.e., an open cover $\mathcal{V}$ with no finite subcover such that for any open $U \notin \mathcal{V}$ the cover $\mathcal{V} \cup \{U\}$ has a finite subcover. Let $\mathcal{W} = \mathcal{V} \cap \mathcal{S}$. Then no finite subfamily of $\mathcal{W}$ covers X (since $\mathcal{V}$ contains no finite subcover). Thus $\mathcal{W}$ is not a cover of X (by (ii)). Let $x \in X \backslash (\cup \mathcal{W})$. Choose a set V in the cover $\mathcal{V}$ with $x \in V$. Since $\mathcal{S}$ generates τ, there are sets $S_1, \ldots, S_n$ in $\mathcal{S}$ with $x \in \bigcap\limits_{i=1}^{n} S_i \subseteq V$. Since $x \notin \cup \mathcal{W}$, no S_i is in $\mathcal{V}$. Since $\mathcal{V}$ is maximal, there exists for each j a set A_j which is the union of a finite number of sets in $\mathcal{V}$ such that $S_j \cup A_j = X$. Then $V \cup \left(\bigcup\limits_{j=1}^{n} A_j \right) \supseteq \left(\bigcap\limits_{j=1}^{n} S_j \right) \cup \left(\bigcup\limits_{j=1}^{n} A_j \right) = \bigcap\limits_{j=1}^{n} \left(S_j \cup \left(\bigcup\limits_{k=1}^{n} A_k \right) \right) \supseteq \bigcap\limits_{j=1}^{n} (S_j \cup A_j) = X$. Thus X is a union of finitely many sets from $\mathcal{V}$, contradicting the choice of $\mathcal{V}$.

Proof of Tychnoff's Theorem: By Alexander's Theorem (the last theorem), it suffices to consider covers of $X = \prod\{X_\alpha : \alpha \in \Lambda\}$ by sets of the form $\pi_\alpha^{-1}(U_\alpha)$ with U_α open in X_α. Let $\mathcal{U}$ be any such cover of X. For each $\alpha \in \Lambda$, let $\mathcal{U}_\alpha$ denote the family of all open sets $U \subseteq X_\alpha$ such that $\pi_\alpha^{-1}(U) \in \mathcal{U}$. We claim that for some α we must have $\cup \mathcal{U}_\alpha = \cup\{U : U \in \mathcal{U}_\alpha\} = X_\alpha$. If not, there would be a point x in X such that for every $\alpha \in \Lambda$, $\pi_\alpha(x) \notin \cup \mathcal{U}_\alpha$. Hence $x \notin \pi_\alpha^{-1}(U)$ for all $\pi_\alpha^{-1}(U) \in \mathcal{U}$, which contradicts the fact that $\mathcal{U}$ is a cover. Thus we may choose $\eta \in \Lambda$ so that $\cup \mathcal{U}_\eta = X_\eta$. Since X_η is compact, there exist $U, \ldots, U_n$ in U_η such that $X_\eta = \bigcup\limits_{i=1}^{n} U_i$. Thus $\{\pi_\eta^{-1}(U_i) : i = 1, \ldots, n\}$ is a finite subcover of $\mathcal{U}$ for X, and the theorem is proved.