Mathematics 6021 Final Examination ñ Summer 2003, W. L. Green

Directions: This is a take-home examination, due July 30, 2003 at 5:40 PM in Skiles 134. Do all problems. Justify your answers. "Show" means "prove." Please put your name on each page of your paper. The symbol $\mathbb{R}$ below will denote the set of all real numbers with the usual topology. This is an open book examination, but you must acknowledge all sources other than your class notes. Cooperation with other students is not permitted. You may, however, discuss the examination with other students provided you write nothing down during the discussion. The definitions and remarks below are for your use in problem 3. There are three pages to this examination.

Definitions: A relation $\leq$ on a set $S$ is \textbf{reflexive} if $x \leq x$ for all $x$ in $S$, and is \textbf{transitive} if $x \leq y$ and $y \leq z$ imply that $x \leq z$. A relation on $S$ directs $S$ if it is symmetric and transitive and has the following additional property: whenever $x$ and $y$ are in $S$, there exists $z$ in $S$ such that $x \leq z$ and $y \leq z$. A \textbf{directed set} is a pair $(S, \leq)$ consisting of a non-empty set $S$ and a relation $\leq$ which directs $S$. A \textbf{net} is a function on a directed set.

Examples: If $S$ is the set of all positive integers, then the usual ordering $\leq$ directs $S$. If $S$ is the set of all subsets (or the set of all finite subsets) of a fixed set $X$, and if $A \leq B$ means that $A$ is contained in $B$, then $\leq$ directs $S$. In particular, a sequence is a net whose domain is the set of all positive integers. As with sequences, we generally write a net defined on a set $S$ as $\{x_s\}$, where $s$ denotes an arbitrary element of the directed set $S$ and $x_s$ denotes the value of the net at $s$. If the net $\{x_s\}$ takes its values in a set $X$, then we say that $\{x_s\}$ is a \textbf{net in $X$}. Suppose now that $X$ is a topological space, that $z$ is an element of $X$, and that $\{x_s\}$ is a net in $X$. We say that $\{x_s\}$ \textbf{converges} to $z$, or that $z$ is a \textbf{limit} of $\{x_s\}$, if for every neighborhood of $z$ there exist an index $u$ such that $u \leq s$ implies $x_s$ is an element of $V$ (i.e., if the net is \textbf{eventually} in every neighborhood of $z$..) We say that $\{x_s\}$ \textbf{clusters} at $z$ if for every neighborhood of $z$ and every index $u$ there exists a $v$ with $u \leq v$ such that $x_v$ is an element of $V$. (Note that $\{x_s\}$ clusters at $z$ if and only if for every neighborhood $V$ of $z$, $\{x_s\}$ \textbf{fails} to be eventually in the complement of $V$.)
1. Let C be a non-void compact convex subset of a real normed vector space X.

   a) Suppose x is an element of X but x is not in C. Show that there exists a hyperplane H which strictly separates x and C.

   b) Show that C is the intersection of all the closed half spaces which contain it.

2. Let K be a convex subset of a real vector space X. A face of K is a convex subset F of K which satisfies the following condition: if x and y are elements of K and t is a real number with 0 < t < 1 and tx + (1 - t)y ∈ F, then x and y must lie in F. (Draw a picture: if a line segment lies in K and meets F in a point which is interior to the line segment, then the whole segment lies in F.) If F is a face of K and F = {x}, then x is called an extreme point of K.

   a) Show that if E is a face of F, and if F is a face of K, then E is a face of K.

   b) Show that the intersection of any collection of faces of K is a face of K.

   c) Suppose K is a non-void compact convex subset of \( \mathbb{R}^n \). Let \( f: \mathbb{R}^n \to \mathbb{R} \) be linear and continuous. (This is actually redundant: every linear map on \( \mathbb{R}^n \) to \( \mathbb{R} \) is continuous.) Let \( s = \sup \{ f(x) : x \in K \} \). Show that \( \{ x \in K : f(x) = s \} \) is a closed face of K.

   d) Let f be a function from a convex set K₁ to a convex set K₂. Then f is affine if f(tx + (1 - t)y) = tf(x) + (1 - t)f(y) for all x and y in K₁ and all real t in [0,1]. Show that if f is continuous and affine from K₁ to K₂, then the inverse image of each closed face of K₂ under f is a closed face of K₁.

   e) Let K be a non-void compact convex subset of a real normed vector space. Show that K has at least one non-void closed face which does not properly contain any other non-void closed face. [Hint: Zorn’s Lemma.]

**Remark:** this minimal non-void face can be shown to be an extreme point.
3. Use the definitions on page 1 above to establish the assertions in parts a) through f) below.

a) Let $X$ be a topological space, let $z$ be a point of $X$, and let $\mathcal{N}_z$ denote the set of all neighborhoods of $z$. Show that $\mathcal{N}_z$ is directed by the relation $\supseteq$, where "$A \supseteq B$" means $B$ is contained in $A$. Show further that any function which assigns to each $V$ in $\mathcal{N}_z$ a point $x(V)$ in $V$ is a net in $X$ which converges to $z$.

b) Let $X$ be a topological space, let $z$ be a point of $X$, and let $W$ be any subset of $X$. Show that $z$ lies in the closure of $W$ if and only if there exists a net with values in $W$ which converges to $z$. (Thus nets allow us to characterize closures in a general topological space in the same way that sequences allow us to characterize closures in a pseudometric space.)

c) Let $f$ be a function from a topological space $X$ to a topological space $Y$. Show that $f$ is continuous if and only if it has the following property: whenever $\{x_s\}$ is a net in $X$ which converges to $z$, then $\{f(x_s)\}$ is a net in $Y$ which converges to $f(z)$.

d) Let $y$ and $z$ be points in a topological space $X$, and let $S$ be the set of all ordered pairs $(A,B)$ consisting of a neighborhood $A$ of $y$ and a neighborhood $B$ of $z$. Show that $S$ is directed by the relation $\leq$, where $(A,B) \leq (A',B')$ means that $A$ contains $A'$ and $B$ contains $B'$.

e) Show that a topological space $X$ is Hausdorff if and only if each net in $X$ converges to at most one point of $X$.

f) Show that if $X$ is a compact topological space, then for every net in $X$ there exists a point of $X$ at which the net clusters. (The converse is also true, but you needn’t prove it.)