Math 6021 Exercise Set 2 — Due June 5, 2003

Recall the following facts about decimal expansions: Let \( n \) be an integer greater than or equal to 2, and for a sequence \( \{x_i\}_{i=1}^n = x \) with values in \( \{0, 1, 2, \ldots, n-1\} \), let

\[
L_n(x) = \sum_{i=1}^{\infty} \frac{x_i}{n^i}
\]

Then \( L_n(x) \) converges to a number in \( [0, 1] \). Moreover, if we exclude sequences which are eventually equal to \( n - 1 \), then every element of \( [0, 1) \) is the sum \( L_n(x) \) for a unique sequence \( x \). Observe that the Cantor ternary set is precisely the collection of all those numbers in \( [0, 1] \) which are either equal to 1 or equal to \( L_3(x) \) for some sequence \( x \) which never takes on the value 1.

**Problem:** Define a function \( f \) from the Cantor ternary set into \( [0, 1] \) as follows:

for a sequence \( \{x_i\}_{i=1}^n \), define a new sequence \( \{y_i\}_{i=1}^n \) by \( y_i = x_i \) if \( x_i \neq 2 \) and \( y_i = 1 \) if \( x_i = 2 \). Then define \( f \) by \( f(1) \) and if \( t = L_3(x) \), then \( f(t) = L_2(y) \).

1) Show that the only connected subsets of the Cantor ternary set are the empty set and singleton sets.

2) Show that \( t \) is a continuous function from the Cantor ternary set onto \( [0, 1] \).

3) If \( f \) had an inverse (it does not), could the inverse be continuous? Why or why not?
Solutions

1) The Cantor set is the intersection (over $n \geq 1$) of sets $K_n$, where each $K_n$ is a union of $2^n$ intervals of length $\frac{1}{3^n}$, each pair of which are separated by some open interval of length $\frac{1}{3^n}$ which is not in $K_n$. Thus no interval in the Cantor set can have length greater than $\inf_{n \geq 0} \frac{1}{3^n} = 0$. But a non-empty connected subset of $[0,1]$ is necessarily an interval, so a connected subset of the Cantor set is an interval of length zero or the empty set.

2) Since $f(1) = 1$ and since every element of $[0,1)$ has a binary expansion of the sort allowed, it follows that every element of $[0,1)$ comes from an element of the Cantor set. (1 comes from 1, and the rest come from allowable ternary expansions using no 1). Thus $f$ is onto. To prove continuity, it suffices to show that each of the following intervals has a relatively open preimage under $f$:

$$
\left( \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{0}{2^n}, \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{1}{2^n} \right)
$$

$$
\left( \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{1}{2^n}, \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_{n-1}+1}{2^{n-1}} \right)
$$

$$
[0, \frac{a_1}{2} + \cdots + \frac{a_n}{2^n})
$$

$$
\left( \frac{a_1}{2} + \cdots + \frac{a_n}{2^n}, 1 \right]
$$

This also follows from the substitution rule in the definition of $f$.

3) If the inverse of $f$ existed and were continuous, then since $[0,1]$ is connected, the Cantor set would be connected. By 1) above, this is not the case.