Mathematics 4317 Final Examination - July 31, 2006

Directions: Do all problems. Show your work, and justify your assertions. Put your name on your bluebook. Closed book. Calculators are allowed. The symbol “$\mathbb{R}$” below denotes the real number system.

1. (20) Let $x$ and $y$ be non-negative real numbers, and let $n$ be a positive integer.
   a) Show that if $x < y$, then $x^n < y^n$. [Hint: factor $x^n - y^n$.]
   b) Show that if $x < y$, then $x^{1/n} < y^{1/n}$. [Hint: Use part a.)]
   c) For non-negative $x$, let $f_n(x) = x^{1/n}$. Recall: except at zero, the sequence $\{f_n(x)\}$ converges to 1, and $\{f_n(0)\}$ converges to zero. Show that if $0 < z < 1$, then the sequence $\{f_n\}$ converges uniformly on $[z,1]$.
   d) Let $\{f_n\}$ be as in part c. Is the convergence of $\{f_n\}$ uniform on $[0,1]$? Why or why not?
   e) Let $\{f_n\}$ be as in part c. Is the convergence of $\{f_n\}$ uniform on $(0,1)$? Why or why not?

2. (8) Use the triangle inequality to prove that for any vectors $x$ and $y$ in $\mathbb{R}^p$, $\|x - y\| \leq \|x\| - \|y\|$. 

3. (12) Show that the set of all polynomial $p(x)$ with integral coefficients is a countable set. Is the same true for the set of all polynomials with rational coefficients? Is it true for the set of all polynomials with real coefficients?

4. (12) a. Show that the intersection of a finite collection of open sets is always open.
   b. Give an example to show that the intersection of infinitely many open sets need not in general be open.

5. (16) Define two functions $p_1$ and $p_2$ on $\mathbb{R}^2$ as follows. For $x = (x_1, x_2)$, let $p_1(x) = x_1$ and $p_2(x) = x_2$. (These functions are called the projections of $\mathbb{R}^2$ onto its coordinate axes.)
   a) Show that $p_1$ and $p_2$ are continuous functions from $\mathbb{R}^2$ into $\mathbb{R}$.
   b) Let $A$ and $B$ be subsets of $\mathbb{R}$, and let $A \times B$ denote the Cartesian product $\{(a,b): a \in A$ and $b \in B\}$ of $A$ and $B$. Show that if $A$ and $B$ are non-empty $A \times B$ is compact, then $A$ and $B$ are compact.
   c) Show that if $A$ and $B$ are closed, then so is $A \times B$.
   d) Show that if $A$ and $B$ are compact, then so is $A \times B$. [Hint: Use the Heine-Borel Theorem.]

6. (16) Suppose the function $f$ takes the interval $[0,1]$ into itself and satisfies $|f(x) - f(y)| \leq \frac{1}{4} |x^2 - y^2|$ for all $x$ and $y$ in this interval.
   a) Show that the image $f([0,1])$ of $[0,1]$ under $f$ is a subinterval of $[0,1]$.
   b) Show that $f$ has exactly one fixed point in the interval $[0,1]$.

7. (16) For each of the following, decide whether the series is convergent or not convergent. You may use what you know from calculus about derivatives and integrals and the log and exponential functions.
   a) $\sum_{n=0}^{\infty} \frac{n}{5^{n+1}}$
   b) $\sum_{n=5}^{\infty} \frac{1}{n(\log n)^2}$
   c) $\sum_{n=1}^{\infty} \left(\frac{n+3}{n}\right)^n$
   d) $\sum_{n=1}^{\infty} \frac{n+3}{n^3 + 27}$

8. (8) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx)$ converges for every real number $x$. Does it converge absolutely?

9. (4) Let $x_n = \left(\frac{1}{n+1}\right)^2 + \left(\frac{1}{n+2}\right)^2 + \ldots + \left(\frac{1}{2n}\right)^2$. Does the sequence $\{x_n\}$ converge? Why or why not?