Directions: Do any four of the five problems below. Show your work, and justify your answers and assertions. This is a closed book examination, and calculators are allowed. Throughout this examination, the symbol “C” will denote the complex number system, and || and < , > will denote norms and inner products. Throughout this examination, “show” means “prove.”

1. (25) Let \( \{x_k\} \) be a sequence in a pre-Hilbert space \( V \).
   a) Use the Cauchy-Schwarz Inequality to show that if \( \| x_k \| \to 0 \), then for all \( z \) in \( V \) we have \( \langle x_k, z \rangle \to 0 \).
   b) Let \( x \) be any element of \( V \). Show that \( \| x_k - x \| \to 0 \) if and only if we have both \( \| x_k \| \to \| x \| \) and \( \langle x_k, x \rangle \to \langle x, x \rangle \).

2. (25) Let \( H \) be the three-dimensional Hilbert space \( \mathbb{C}^3 \) with the usual inner product. Let \( S \) be the subset \( \{ (1, 1, 0), (1, 1, i), (1, 1, -i) \} \) of \( \mathbb{C}^3 \), where \( i \) denotes the usual square root of \(-1\).
   a) Find an orthonormal set in \( \mathbb{C}^3 \) with the same span as \( S \).
   b) Is the set \( S \) a total subset of the space \( \mathbb{C}^3 \)? Why or why not?

3. (25) Let \( S \) be a linear subspace of a Hilbert space \( H \). Prove that \( S \) is complete if and only if \( S \) is closed. (Your text gives a proof of this result for a subset \( S \) of a complete metric space \( H \); I want to see a similar proof for the case where \( H \) is a Hilbert space and \( S \) is a linear subspace.)

4. (25) Let \( H \) be a Hilbert space, let \( C \) be a non-empty subset of \( H \), and let \( x \) be a point in \( H \). Recall that if \( C \) is closed and convex, then there exists a unique point of \( C \) that is nearest to \( x \).
   a) Give an example to show that if \( C \) is not closed, then there may fail to be a nearest point in \( C \) to \( x \).
   b) Give an example to show that if \( C \) is not convex, then a nearest point to \( x \) in \( C \) may fail to be unique.

5. (25) Let \( \{x_k\} \) be a bounded orthogonal sequence of non-zero vectors in a pre-Hilbert space \( V \), and let \( S = \sup \{ \| x_k \| : 1 \leq k \} \).
   a) Show that for every \( x \) in \( V \), we have \( \sum_{k=1}^{\infty} |\langle x, x_k \rangle|^2 \leq \| x \|^2 S^2 \).
   b) Show that for every \( x \) in \( V \), the sequence \( \{ \langle x_k, x \rangle \} \) converges to zero.
1a) $\langle x, y \rangle \leq \|x\| \cdot \|y\|$, and $\|x_n\| \to 0$. Thus $\langle x, y \rangle \to 0$.

b) Suppose $\|x_n - x\| \to 0$. Then $0 \leq \|x_n - y\| \leq \|x_n - x\| + \|x - y\| \to 0$, so $\|x_n - y\| \to 0$. Also $\langle x_n - x, y \rangle \to 0$ by part a). Thus $\langle x, y \rangle = \langle x, \tilde{y} \rangle \to 0$, so $\langle x, y \rangle \to 0$.

Suppose $\|x_n - 0\| \to 0$ and $\langle x_n, y \rangle \to \langle x, y \rangle$. Then $\|x_n\|^2 \to 0$, and $\langle x_n, y \rangle = \langle x_n, y \rangle + \langle x, y \rangle$.

$\langle x_n, y \rangle = \langle x_n - x, x_n - x \rangle \\
\|x_n - x\|^2 = \langle x_n, x \rangle - \langle x, x \rangle$

$\|y\|^2 - \langle y, x \rangle = \|x\|^2 - \langle x, x \rangle = 0$

2. First normalize $(1, 1, 0)$ to get $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$. We apply Gram-Schmidt:

$\begin{align*}
(1, 1, 0) - \langle 1, 1, 0, 1, 1, 0 \rangle (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = (1, 1, 0) - \sqrt{2} (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) = (0, 0, 0)
\end{align*}$

Since $\langle 0, 0, 0, 0 \rangle = 0$, the first two elements of our orthonormal set are $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ and $(0, 0, 0)$. Observe that the original set $S$ is linearly dependent, since $\frac{1}{2} (1, 1, 0) + \frac{1}{2} (1, 1, -i) = (1, 1, 0)$. Thus the span of $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$ and $(0, 0, 0)$ agrees with the span of $S$.

3. Suppose $S$ is closed in $H$. Let $(x_n)$ be Cauchy in $S$. Then $(x_n)$ is a Cauchy sequence in $H$. Since $H$ is complete, we have $x_n \to x$ for some $x \in H$. Since $S$ is closed and $x_n \to x$ with $x_n \in S$, we have $x \in S$. Then every Cauchy sequence in $S$ converges to a point of $S$, so $S$ is complete.

Suppose $S$ is complete. Let $(x_n)$ with $x_n \to x \in H$. Since $(x_n)$ converges in $H$, it is a Cauchy sequence in $H$. Hence it is a Cauchy sequence in $S$. Since $S$ is complete, we have $x_n \to x$ for some $x \in S$. Thus $x_n \to x$ and $x_n \to 0$.

By uniqueness of limits, $x = 0$, so $x \in S$. Thus $S$ is closed.

4a) Let $H = \mathbb{C}$. Let $x = 0$, and let $C = (0, i)$ (on the real axis).

b) Let $H = \mathbb{C}$. Let $x = 0$. Let $C$ be the circle of radius 1 centered at zero.
5c) \[ \sum_{k} |\langle x, x_k \rangle|^2 = \sum_{k} \frac{x_k}{\|x_k\|} \sum_{k} \frac{x_k}{\|x_k\|} \leq \sum_{k} \frac{x_k}{\|x_k\|} \|x_k\|^2 \leq S^2 \sum_{k} \frac{x_k}{\|x_k\|} \|x_k\|^2 \]
\[ \leq S^2 \|x\|^2, \text{ by Bessel's inequality} \]

b) Since \[ \sum_{k} |\langle x, x_k \rangle|^2 \] converges, \[ |\langle x, x_k \rangle|^2 \to 0 \] as \[ \langle x, x_k \rangle \to 0 \].