Mathematics 4317 Final Examination - July 30, 2009

Directions: Do any eight of the ten problems below. If you attempt more than eight, indicate on the front of your bluebook which set of eight you want graded. Show your work, and justify your assertions. This is a closed book examination, and calculators are allowed. The symbol “R” below will denote the real number system. Please don’t forget to put your name on your bluebook; it is worth four points.

1. (12) Let \( f \) be a function from a set \( X \) to a set \( Y \), and let \( T \) and \( V \) be subsets of \( Y \). Show that \( f^{-1}(T \cap V) = f^{-1}(T) \cap f^{-1}(V) \).

2. (12) Use the supremum property of the real numbers to show that if \( S \) is a non-empty subset of \( \mathbb{R} \), and if \( S \) is bounded below, then \( S \) has an infimum (i.e., a greatest lower bound).

3. (12) The diameter of a bounded set \( S \) is the number \( d(S) = \sup\{|x - y| : x \in S \text{ and } y \in S\} \). Suppose \( S_n \) is a nested sequence of sets such that \( d(S_n) < \frac{1}{n} \) for all \( n \). Give a proof or a counterexample to the following assertion: \( \bigcap_{n=1}^{\infty} S_n \) is a single point.

4. (12) Show that a subset \( K \) of \( \mathbb{R}^p \) is compact if and only if it has the following property: if \( \{x_n\} \) is a sequence in \( K \), then there exists a subsequence of \( \{x_n\} \) that converges to a point of \( K \).

5. (6) a) Show that the sequence \( \sqrt{n+1} - \sqrt{n} \) converges to zero.
   b) Recall that the sequence \( \left(1 + \frac{1}{n}\right)^n \) converges to a number that we call \( e \), with \( e \) between 2 and 3. Show that the sequence \( \left(1 + \frac{1}{2n}\right)^n \) converges to \( \sqrt{e} \).

6. (12) Let \( K \) be a compact convex subset of \( \mathbb{R}^p \), and let \( f : K \to \mathbb{R} \) be continuous. Show that \( f(K) \) is a closed bounded interval in \( \mathbb{R} \). [Recall that a set \( S \) is convex if and only if it has the following property: whenever \( x \) and \( y \) are in \( S \) and \( t \) is a real number between 0 and 1, then \( tx + (1-t)y \) is also in \( S \).]

7. Let \( f_n(x) = \frac{x^n}{1 + x^n} \) for \( x > 0 \).
   a) (4) Show that \( f_n \) converges to 1 on \((1, +\infty)\), and that \( f_n \) converges to zero on \((0, 1)\).
   b) (4) Is the convergence in part a) of this problem uniform on either of the sets \((1, +\infty)\) or \((0, 1)\)? Give a reason for your answer.
   c) (4) Let \( f = \lim f_n \). Is \( f \) continuous on \([1, +\infty)\)? Why or why not?

8. Suppose the function \( f \) takes the interval \([0, 1]\) into itself and satisfies \( |f(x) - f(y)| \leq \frac{1}{4} |x^2 - y^2| \) for all \( x \) and \( y \) in this interval.
   a) (6) Show that \( f \) is continuous on \([0, 1]\). Is it uniformly continuous? Why or why not? [Hint: factor \( x^2 - y^2 \).
   b) (6) Show that \( f \) has exactly one fixed point in the interval \([0, 1]\).

[over]
9. Find the radius of convergence of each of the following.

a) \( \sum_{n=0}^{\infty} \frac{n}{7^n} x^n \)

b) \( \sum_{n=0}^{\infty} \frac{n!}{7^n} x^n \)

c) \( \sum_{n=1}^{\infty} \frac{(n + 3)^2}{(n + 1)^3} x^n \).

10a) (4) Show that the series \( \sum_{n=1}^{\infty} \frac{1}{n^7} x^n \) converges at -7 and at -2. At each of these points, does it converge absolutely or conditionally? Give reasons for your answers.

b) (4) Show that the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(nx) \) converges for every real number \( x \). Is the convergence uniform on \( \mathbb{R} \)?

c) (4) Show that if \( f \) is an odd function, then all of the Fourier coefficients \( a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \) are zero.
Let $x \in f^{-1}(TNV)$, then $f(x) \in TNV$, so $f(x) \in T$ and $f(x) \in V$.

Thus, $x \in f^{-1}(T)$ and $x \in f^{-1}(V)$, so $x \in f^{-1}(T) \cap f^{-1}(V)$.

Let $x \in f^{-1}(T) \cap f^{-1}(V)$. Then $x \in f^{-1}(T)$ and $x \in f^{-1}(V)$, so $f(x) \in T$ and $f(x) \in V$, so $f(x) \in TNV$, so $x \in f^{-1}(TNV)$.

Let $S$ be non-empty and bounded below by $-2$. Then the set $-S = \{ -s : s \in S \}$ is non-empty and bounded above by $2$. Then $-S$ has a least upper bound, say $u$. Then $-u$ is a lower bound for $S$. We claim $-u$ is the greatest lower bound. If $b$ is any lower bound for $S$, then $a > -b$ is an upper bound for $-S$, so $-u \leq -b$, so $u \leq b$.

3. The assertion is false. Let $S_n = (0, \frac{1}{n})$. Then $S_n \supset S_n+1$ for all $n$, and $d(S_n) = \frac{1}{2n} \leq \frac{1}{2n}$ for each $n$. However, $\bigcup S_n$ is empty.

(The assertion would be correct if the $S_n$ were all closed.)

4. Let $K$ be compact. Let $(x_n)$ be a sequence in $K$. Since $K$ is bounded, $(x_n)$ has a convergent subsequence, which also lies in $K$. Since $K$ is closed, the limit of this subsequence lies in $K$. Conversely, suppose $K$ has the stated property. If $K$ is not compact, then either $K$ is not bounded or $K$ is not closed. If $K$ is unbounded, choose $x_n \in K$ with $|x_n| > n$.

Then $(x_n)$ is unbounded, and so every subsequence of $(x_n)$, such as $(x_{2n})$, has $x_{2n} \in K$ for all $n$. This is a contradiction. If $K$ is not closed, then let $x$ be a boundary point of $K$ such that $x \notin K$. Choose a sequence $(x_n)$ in $K$ such that $|x_n - x| < \frac{1}{n}$ for all $n$. Then $x_n \rightarrow x$. But $x \notin K$.

so we again have a contradiction. Thus $K$ is compact.

5. $\lim \sqrt{n+1} - \sqrt{n} = \lim \frac{(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \cdot (\sqrt{n+1} - \sqrt{n}) = \frac{1}{\sqrt{n+1} + \sqrt{n}}$.

$\leq \frac{1}{2\sqrt{n}} \rightarrow 0$. Thus $\sqrt{n+1} - \sqrt{n} \rightarrow 0$.

6. (b) $(1 + \frac{1}{2n})^n = \left(1 + \frac{1}{2n}\right)^{2n} \rightarrow e^{\frac{1}{2}} = \sqrt{e}$, since $(1 + \frac{1}{2n})^{2n}$ is a subsequence of $(1 + \frac{1}{n})^n$ and the sequence $\ln$ is a continuous function on $[0, +\infty)$. 6. Convex sets are polygonally connected hence connected. Since $f$ is continuous...
on \( K \) and preserves convexity and connectedness, \( f(K) \) is convex and connected, i.e., is a closed bounded interval.

7a) \( f_n(x) = \frac{x^n}{1+x^n} \to \frac{x}{x+1} = 1 \quad \text{for} \quad x > 1 \), while

\[
0 \leq f_n(x) = \frac{x^n}{1+x^n} < x^n \to 0 \text{ if } 0 < x < 1 \text{, and } \lim_{n \to \infty} x^n = 0 \quad \text{if } x < 0.
\]

b) Let \( x > 1 \). Then \( 1 - x^n = \frac{1}{1+x^n} = \frac{1}{1+x} \) and \( \lim_{n \to \infty} \frac{1}{1+x^n} = 1 \quad \text{for} \quad 1 < x \).

Since \( \lim_{n \to \infty} f_n(x) = 0 \), the convergence is uniform on \( [0, \infty) \).

b) Let \( 0 < x < 1 \). Then \( \frac{x^n}{1+x^n} = \frac{1}{1+x} \) has supremum \( \frac{1}{x+1} \),

so \( \lim_{n \to \infty} f_n(x) = 0 \), so the convergence is not uniform on \( (0, 1) \).

c) \( f(1) = \frac{1}{2} \) and \( f(x) < 1 \quad \text{for} \quad 1 < x \), so \( f \) is not continuous at \( 1 \).

8a) \( \|f(g)\| \leq \frac{1}{4} \|x^2-y^2\| = \frac{1}{4} (x+y)(x-y) \leq \frac{1}{4} (2) \|x-y\| = \frac{1}{2} \|x-y\|.

Thus \( f \) is a strict contraction, in particular it is uniformly continuous.

b) Since \( g \) maps \([0, 1]\) into itself and is a strict contraction, the proof of the fixed point theorem for contractions is valid for \( f \). Thus \( f \) has a unique fixed point in \([0, 1]\).

g) \( \frac{m+1}{m^2} \to \frac{1}{m} \to 1 \), so the radius of convergence is 1.

b) \( \frac{m+1}{m^2} \to \frac{m+1}{m^2} \to 1 \to 0 \), so the radius of convergence is 0.

\[
\left( \frac{m+1}{m} \right)^{\frac{m}{n}} = \left( \frac{m+1}{m} \right)^{\frac{m}{n}} \to 1 \quad \text{as } \frac{m}{n} \to 0 \quad \text{and} \quad e^3 \to e^3.
\]

Thus the radius of convergence is \( \frac{1}{e} \).

10a) \( \sum (-1)^n \) converges by the alternating series test.

The convergence is conditional, since \( 2^n \) diverges. At \(-2\), we get \( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \)

which converges absolutely by the ratio or root test.

b) Since \( |a_n| = \frac{1}{n^2} \leq \frac{1}{n^2} \) and \( \sum \frac{1}{n^2} \) converges, the series converges absolutely and

uniquely by Weierstrass' M-test.

c) By Taylor's Theorem, \( f(x) = \sum_{n=0}^{\infty} a_n (x-2)^n \) at \( x = 2 \).