Directions: Do any four of the five problems below. If you do more that four, indicate on your paper which four problems you want to have graded. Show your work, and justify your answers and assertions. This is a closed book examination, and calculators are allowed. Throughout this examination, the symbol “R” will denote the real number system, and || || and \langle, \rangle will denote the usual norm and inner product on \mathbb{R}^p.

1.
   a) (10) Give an explicit bijection between the set of all positive integers and the set of all integers.
   b) (10) A rational polynomial of degree n is a function of the form \( p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \), where n is a non-negative integer, where \( a_n \) is non-zero and where each \( a_i \) is a rational number. Let \( S_n \) be the set of all rational polynomials of degree n. Is the set \( S_n \) finite, countably infinite, or uncountable? Give a proof of the assertion in your answer.
   c) (5) Let \( S \) be the union of all the sets \( S_n \), where n ranges over the non-negative integers. Is the set \( S \) finite, countably infinite or uncountable? Give a reason for your answer.

2.
   a) (10) State carefully either the Supremum Property or the Archimedean Property.
   b) (10) Let \( S \) be a subset of \( \mathbb{R} \). Show that if \( S \) has a supremum, then that supremum is a boundary point of \( S \).
   c) (5) Let \( S \) be a closed bounded subset of \( \mathbb{R} \). Show that the supremum of \( S \) is an element of \( S \).

3. Use the properties and definitions of the norm and inner product in \( \mathbb{R}^p \) to prove the following:
   a) (10) For all \( x \) and \( y \) in \( \mathbb{R}^p \), \( \|x\|^2 + \|y\|^2 = \|x+y\|^2 \) if and only if \( \langle x, y \rangle = 0 \). (This assertion includes the Pythagorean theorem.)
   b) (5) If \( n(x, y) = 6|y| - 6|x| \), then \( n \) is not a norm on \( \mathbb{R}^2 \).
   c) (10) Show that for all \( x = (x_1, x_2, \ldots, x_p) \) in \( \mathbb{R}^p \), we have \( \|x\| \leq \sqrt{p} \). [Hint: Use the Cauchy-Schwarz Inequality.]

4.
   a) (10) State carefully either the Heine-Borel Theorem or the Bolzano-Weierstrass Theorem.
   b) (10) Let \( F \) and \( K \) be compact subsets of \( \mathbb{R}^2 \). Show that the union of \( F \) and \( K \) is also compact.
   c) (5) Let \( K \) be a compact subset of \( \mathbb{R}^2 \), and let \( F \) be a closed subset of \( \mathbb{R}^2 \). Show that the intersection of \( F \) and \( K \) is also compact.

5.
   a) (15) Let \( A \) and \( B \) be two open connected subsets of \( \mathbb{R}^p \), and suppose that the intersection of \( A \) and \( B \) is not empty. Is the union of \( A \) and \( B \) connected? Give a reason for your answer.
   b) (10) Suppose \( A \) and \( B \) are as in part a) above, and let \( U \) be the union of \( A \) and \( B \). Let \( x \) and \( y \) be points of the union \( U \). Can we always join \( x \) to \( y \) by a polygonal curve that lies wholly in \( U \)? Give a proof or a counterexample.
1a) Let \( f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases} \) (Many other solutions are possible.)
b) We claim that \( S_n \) is countably infinite for each \( n \). Now \( S_n \) is infinite for each \( n \), since it always contains the set of all non-zero constant rational polynomials, which is in one-to-one correspondence with the set of all non-zero rationals, which is infinite. It is enough then to show that the collection \( T_n \) of all polynomial \( a_0 + a_1x + a_2x^2 + \ldots + a_n x^n \) (including the ones where \( a_n \) is zero) is countable for each \( n \), since \( T_n \) contains the set \( S_n \), and a countable subset of a countable set is again countable. We prove by induction that \( T_n \) is countable. If \( n = 0 \), then \( T_n \) is in one-to-one correspondence with the rationals, and so is countable. Suppose then that \( T_n \) is countable, and consider \( T_{n+1} \). Now each element \( a_0 + a_1x + a_2x^2 + \ldots + a_n x^n \) of \( T_{n+1} \) is determined by the polynomial \( a_0 + a_1x + a_2x^2 + \ldots + a_n x^n \) in \( T_n \) and by the rational number \( a_{n+1} \). Thus the set \( T_{n+1} \) is a countable union of copies (one for each rational number \( a_{n+1} \)) of \( T_n \). But a countable union of countable sets is countable.
c) \( S \) is the countable union of all the sets \( S_n \), each of which is countable. Thus \( S \) is countable.

2a) See your text.
b) Let \( u \) be the supremum of \( S \), and let \( e > 0 \) be given. Since \( u - e < u \) and \( u \) is the least upper bound of \( S \), there exists an element of \( S \) that is strictly greater than \( u - e \) (and less than or equal to \( u \)). Since \( u + e > u \) and \( u \) is an upper bound of \( S \), no element of \( (u, u + e) \) lies in \( S \), i.e., every element of \( (u, u + e) \) lies in the complement of \( S \). Thus every neighborhood of \( u \) contains points of \( S \) and points of the complement of \( S \).
c) A closed set contains all of its boundary points.

3a) \( ||x||^2 + ||y||^2 = xx + yy \) and \( ||x + y||^2 = (x + y)(x + y) = xx + 2xy + yy \). Thus \( ||x||^2 + ||y||^2 = ||x + y||^2 \) if and only if \( xy = 0 \).
b) Many solutions are possible; for example, \( n(1, 1) = 0 \), but \( (1, 1) \) is not the zero vector.
c) \( ||x_1|| + ||x_2|| + \ldots + ||x_p|| = (||x_1||, ||x_2||, \ldots, ||x_p||) \cdot (1, 1, \ldots, 1) \leq ||x||^{p/2} \) (since \( (x_1, x_2, \ldots, x_p) \) and \( (||x_1||, ||x_2||, \ldots, ||x_p||) \) have the same norm, and since the norm of \( (1, 1, \ldots, 1) \) is \( p^{1/2} \)).

4a) See your text.
b) Let \( C \) be an open cover for the union of \( F \) and \( K \). Then \( C \) covers \( F \). Since \( F \) is compact, there is a finite subcover (for \( F \)) of \( C \). Similarly, since \( K \) is compact and \( C \) covers \( K \), there is a finite subcover (for \( K \)) of \( C \). The union of these two subcovers is a finite subcover for \( C \) for the union of \( F \) and \( K \). (Or note that \( F \) and \( K \) are closed and bounded, so their union is closed and bounded, hence compact.)
c) The intersection of \( F \) and \( K \) is a closed subset of the bounded set \( K \). Thus the intersection is closed and bounded, hence compact.

5a) Since \( A \) and \( B \) are open and connected, \( A \) and \( B \) are polygonally connected. We claim that the union \( U \) of \( A \) and \( B \) is polygonally connected, and hence connected. Let \( x \) and \( y \) lie in \( U \). If \( x \) and \( y \) lie in the same one of \( A \) or \( B \), then they can be connected by a polygonal curve contained in \( A \) or \( B \), which curve is also a polygonal curve contained in \( U \). We may assume then that \( x \) and \( y \) lie in \( A \) and \( B \) respectively. Let \( z \) be any point in the intersection of \( A \) and \( B \). Connect \( x \) to \( z \) by a polygonal curve contained in \( A \), and connect \( z \) to \( y \) by a polygonal curve contained in \( B \). The union of these two curves is a polygonal curve contained in \( U \) that connects \( x \) to \( y \). Since \( x \) and \( y \) are arbitrary points of \( U \), \( U \) is polygonally connected, as we claimed.
b) See part a) above.