Final solutions (blue version)

```
> with(linalg);
Warning, the protected names norm and trace have been redefined and
unprotected
```

(1)

[*BlockDiagonal*, *GramSchmidt*, *JordanBlock*, *LUdecomp*, *QRdecomp*, *Wronskian*, *addcol*, *addrow*, *adj*, *adjoint*, *angle*, *augment*, *backsub*, *band*, *basis*, *bezout*, *blockmatrix*, *charmat*, *charpoly*, *cholesky*, *col*, *coldim*, *colspace*, *colspan*, *companion*, *concat*, *cond*, *copyinto*, *crossprod*, *curl*, *definite*, *delcols*, *delrows*, *det*, *diag*, *diverge*, *dotprod*, *eigenvals*, *eigenvalues*, *eigenvectors*, *eigenvects*, *entermatrix*, *equal*, *exponential*, *extend*, *ffgausselim*, *fibonacci*, *forwardsub*, *frobenius*, *gausselim*, *gaussjord*, *geneqns*, *genmatrix*, *grad*, *hadamard*, *hermite*, *hessian*, *hilbert*, *htranspose*, *ihermite*, *indexfunc*, *innerprod*, *intbasis*, *inverse*, *ismith*, *issimilar*, *iszero*, *jacobian*, *jordan*, *kernel*, *laplacian*, *leastsqrs*, *linsolve*, *matadd*, *matrix*, *minor*, *minpoly*, *mulcol*, *mulrow*, *multiply*, *norm*, *normalize*, *nullspace*, *orthog*, *permanent*, *pivot*, *potential*, *randmatrix*, *randvector*, *rank*, *ratform*, *row*, *rowdim*, *rowspace*, *rowspan*, *rref*, *scalarmul*, *singularvals*, *smith*, *stackmatrix*, *submatrix*, *subvector*, *sumbasis*, *swapcol*, *swaprow*, *sylvester*, *toeplitz*, *trace*, *transpose*, *vandermonde*, *vecpotent*, *vectdim*, *vector*, *wronskian*]

1. (10 points) In this problem, S designates the surface

x^2 + y^2 = 4 z^4 .

(This surface is what you get by rotating the parabolic arc {y=0, x=2 z^2} around the z-axis.)

a) Describe this surface in terms of an angular coordinate theta and another reasonable coordinate, u.

```
> xcomp := (theta, u) -> u*cos(theta); ycomp := (theta, u) -> u*
sin(theta); zcomp := (theta, u) -> sqrt(u/2);
```

$$xcomp := (\theta, u) \to u\cos(\theta) \quad (2)$$

$$ycomp := (\theta, u) \to u\sin(\theta)$$

$$zcomp := (\theta, u) \to \sqrt{\frac{1}{2}u}$$

b) In terms of these coordinates, find a normal vector to the surface:

```
> vector([diff(xcomp(theta,u),theta), diff(ycomp(theta,u),theta),
diff(zcomp(theta,u),theta)]);
```

$$\begin{bmatrix} -u\sin(\theta) & u\cos(\theta) & 0 \end{bmatrix} \quad (3)$$

```
> r_theta := unapply(%,theta,u);
```

r_theta := (θ, *u*) (4)

→*table([(1) = `+`(`-`(`*`(u, `*`(sin(theta))))), (2) = `*`(u, `*`(cos(theta))), (3) = 0])*

```
> vector([diff(xcomp(theta,u),u), diff(ycomp(theta,u),u), diff
  (zcomp(theta,u),u)]);
```

$$\left[\cos(\theta) \quad \sin(\theta) \quad \frac{1}{4}\frac{\sqrt{2}}{\sqrt{u}}\right] \tag{5}$$

```
> r_u := unapply(%,theta,u);
```

$$r\_u := (\theta, u) \to table(\,[(\,1\,) = cos(theta),\ (\,2\,) = sin(theta),\ (\,3\,) = `+`(`/`(`*`(`/`(1, 4),\ `*`(`*`(sqrt(2)))),\ `*`(`*`(sqrt(u))))\,]\,) \tag{6}$$

```
> crossprod(r_theta(theta,u), r_u(theta,u));
```

$$\left[\frac{1}{4}\sqrt{u}\cos(\theta)\sqrt{2} \quad \frac{1}{4}\sqrt{u}\sin(\theta)\sqrt{2} \quad -u\sin(\theta)^2 - u\cos(\theta)^2\right] \tag{7}$$

This simplifies to

```
> N := (theta, u) -> vector([(1/(2*sqrt(2)))*sqrt(u)*cos(theta),
  (1/(2*sqrt(2)))*sqrt(u)*sin(theta), -u]);
```

$$N := (\theta, u) \to \left[\frac{1}{2}\frac{\sqrt{u}\cos(\theta)}{\sqrt{2}} \quad \frac{1}{2}\frac{\sqrt{u}\sin(\theta)}{\sqrt{2}} \quad -u\right] \tag{8}$$

c) This vector points to the outside of the surface and has length

```
> J := (theta, u) -> sqrt(u^2 + u/8);
```

$$J := (\theta, u) \to \sqrt{u^2 + \frac{1}{8}u} \tag{9}$$

d) In terms of the Cartesian coordinates (x,y,z), find a normal vector to the surface:

One way to do this is to convert from cylindrical to Cartesian - u and theta are the same as the polar coordinates. Since the three coordinates are related, more than one correct answer is possible. For instance:

```
> vector([x/(2*sqrt(2)*(x^2+y^2)^(1/4)), y/(2*sqrt(2)*(x^2+y^2)^
  (1/4)), - sqrt(x^2+y^2)]);
```

$$\left[\frac{1}{4}\frac{x\sqrt{2}}{(x^2+y^2)^{1/4}} \quad \frac{1}{4}\frac{y\sqrt{2}}{(x^2+y^2)^{1/4}} \quad -\sqrt{x^2+y^2}\right] \tag{10}$$

e) Calculate the area of S between height z= 1 and z = 2

Surface area =

```
> 2*Pi*integrate(J(theta, u), u=2..8);
```

$$2\pi\left(\frac{129}{32}\sqrt{65} - \frac{1}{512}\ln\left(129 + 16\sqrt{65}\right) - \frac{33}{64}\sqrt{17} + \frac{1}{512}\ln\left(33 + 8\sqrt{17}\right)\right) \tag{11}$$

```
> evalf(%);
```

$$190.8350222 \tag{12}$$

2 (10 points) Find the maximum value of f(x,y,z) = 3 x^2 y^2 z^2, given that

x^2 + 4 y^2 + 9 z^2 = 27.

This problem may be solved with Lagrange's method.

```
> f := (x,y,z) -> 3*x^2 * y^2 * z^2; g := (x,y,z) -> x^2 + 4* y^2
+ 9*z^2;
```

$$f := (x, y, z) \rightarrow 3\, x^2\, y^2\, z^2 \quad \textbf{(13)}$$

$$g := (x, y, z) \rightarrow x^2 + 4\, y^2 + 9\, z^2$$

```
> grad(f(x,y,z), vector([x,y,z]));grad(g(x,y,z), vector([x,y,z]));
```

$$\left[\; 6\, x\, y^2\, z^2 \quad 6\, x^2\, y\, z^2 \quad 6\, x^2\, y^2\, z \;\right] \quad \textbf{(14)}$$

$$\left[\; 2\, x \quad 8\, y \quad 18\, z \;\right]$$

From Lagrange's formula that grad f = lambda grad g, we soon find that x = (+/-) 3, y = (+/-) 3/2, and z = (+/-)1. Any and all of these points produce the maximum value of f, which is

```
> f(3,3/2,1); evalf(%);
```

$$\frac{243}{4} \quad \textbf{(15)}$$

$$60.75000000$$

3. (10 points) Let Ω be the solid surrounded by the surface S from Problem 1, between height z=1 and z=2. Find the volume of Ω and its centroid as follows:

a, b) Set up an integral for the volume explicitly in any two of the 3-dimensional coordinate systems we have used in class. For example, in the cylindrical system, the volume would be

```
> Int(Int(Int(r, r=0..2*z^2), z=1..2), theta=0..2*Pi);
```

$$\int_0^{2\pi}\int_1^{2}\int_0^{2z^2} r\, \mathrm{d}r\, \mathrm{d}z\, \mathrm{d}\theta \quad \textbf{(16)}$$

In the Cartesian system, we could write

```
> Int(Int(Int(1, x=-sqrt(4*z^4-y^2)..sqrt(4*z^4-y^2)), y=-2*z^2.
.2*z^2), z=1..2);
```

$$\int_1^{2}\int_{-2z^2}^{2z^2}\int_{-\sqrt{4z^4-y^2}}^{\sqrt{4z^4-y^2}} 1\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \quad \textbf{(17)}$$

c) Evaluate the integral:

```
> 2*Pi*int(int(r, r=0..2*z^2), z=1..2); evalf(%);
```

$$\frac{124}{5}\pi \quad \textbf{(18)}$$

$$77.91149782$$

d) Find the centroid of the figure. (Hint: only one more integral is necessary. Why?)
The x and y components are 0 by symmetry. As for the z-component, it is

```
> 2*Pi*int(int(z*r, r=0..2*z^2), z=1..2)/(124*Pi/5); evalf(%);
```

$$\frac{105}{62} \quad \textbf{(19)}$$

$$1.693548387$$

4. (10 points) In this problem you are to find the total flux of the vector field

```
> v := vector([4*x*y, y^2,z^3]);
```

$$v := \left[\begin{array}{ccc} 4\,x\,y & y^2 & z^3 \end{array}\right] \tag{20}$$

out of the ball of radius 3 centered at the origin.

a) Write the total flux as a specific surface integral using two of the coordinate systems: Cartesian, spherical, or cylindrical:

The normal vector to the sphere is the unit radial vector, so the integrand is

```
> dotprod(v, vector([x/sqrt(x^2+y^2+z^2), y/sqrt(x^2+y^2+z^2),
  z/sqrt(x^2+y^2+z^2)]));
```

$$4\,x\,y\,\frac{x}{\sqrt{x^2+y^2+z^2}} + y^2\,\frac{y}{\sqrt{x^2+y^2+z^2}} + z^3\,\frac{z}{\sqrt{x^2+y^2+z^2}} \tag{21}$$

In spherical coordinates, it is

```
> 4*rho^2*sin(phi)^3 *cos(theta)^2*sin(theta) + rho^2*sin(phi)^3 *
  sin(theta)^3 + rho^3*cos(phi)^4;
```

$$4\,\rho^2\sin(\phi)^3\cos(\theta)^2\sin(\theta) + \rho^2\sin(\phi)^3\sin(\theta)^3 + \rho^3\cos(\phi)^4 \tag{22}$$

while in cylindrical coordinates it is

```
> 4*r^3 *cos(theta)^2*sin(theta)/sqrt(r^2+z^2) + r^3 *sin(theta)
  ^3/sqrt(r^2+z^2) + z^4/sqrt(r^2+z^2);
```

$$\frac{4\,r^3\cos(\theta)^2\sin(\theta)}{\sqrt{r^2+z^2}} + \frac{r^3\sin(\theta)^3}{\sqrt{r^2+z^2}} + \frac{z^4}{\sqrt{r^2+z^2}} \tag{23}$$

To set up the integrals as surface integrals, it may be best to separate the upper and lower hemispheres. Then:

In the Cartesian system, each integral is of the form

```
> Int(Int(integrand * sqrt(9/(9-x^2-y^2)), x = -sqrt(9-y^2)..sqrt
  (9-y^2)), y=-3..3);
```

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} 3\,integrand\,\sqrt{\frac{1}{9-x^2-y^2}}\,\mathrm{d}x\,\mathrm{d}y \tag{24}$$

In the polar or cylindrical system, each integral is of the form

```
> Int(Int(integrand * sqrt(9/(9-r^2))*r, r = 0..3), theta=0..2*Pi)
  ;
```

$$\int_{0}^{2\pi}\int_{0}^{3} 3\,integrand\,\sqrt{\frac{1}{9-r^2}}\,r\,\mathrm{d}r\,\mathrm{d}\theta \tag{25}$$

In the spherical system, we do not need to break the integral into 2 parts. We write

```
> Int(Int(integrand * 3^2*cos(phi), phi=0..Pi), theta=0..2*Pi);
```

$$\int_0^{2\pi}\int_0^{\pi} 9\, integrand \cos(\phi)\, d\phi\, d\theta \qquad (26)$$

To evaluate the flux it is simplest to use the divergence theorem

```
> diverge(vector([4*x*y, y^2, z^3]), vector([x,y,z]));
```

$$6y + 3z^2 \qquad (27)$$

This should be integrated over the sphere. Actually, the 6 y is an odd function, so it does not contribute. Using spherical coordinates, we get

```
> 2*Pi*int(int(3*(rho*cos(phi))^2*rho^2*sin(phi), phi=0..Pi), rho
  = 0..3); evalf(%);
```

$$\frac{972}{5}\pi \qquad (28)$$

$$610.7256119$$

5. (10 points) In this problem, the curve C runs counterclockwise around the edge of the triangle connecting the points: (3,1), (5, 4), (-1,2)

a) Evaluate the line integral of a terrible-looking integrand. However, Q_x - P_y = 2, so the integral is twice the area of the triangle. But twice the triangle is a parallelogram, so the answer is the magnitude of

```
> crossprod(vector([5-3,4-1,0]),vector([-1-5,2-4,0]));
```

$$\begin{bmatrix} 0 & 0 & 14 \end{bmatrix} \qquad (29)$$

That is, 14.

b) Evaluate the line integral of F * dr, where the vector field is the gradient of something. Answer: 0, because the gradient of a function is conservative.

6 (10 points). At time t an object is at position

```
> r := t -> vector([4 +  t^2, 1 + 2* t^3, 2 *t^3]);
```

$$r := t \rightarrow \begin{bmatrix} 4+t^2 & 1+2t^3 & 2t^3 \end{bmatrix} \qquad (30)$$

a) A specific integral for the length of the curve from the plane z=0 to the plane z=16 is:

```
> Int(sqrt((2*t)^2+(6*t^2)^2+(6*t^2)^2), t=0..2);
```

$$\int_0^2 2\sqrt{t^2+18t^4}\, dt \qquad (31)$$

b) The length of the curve between the plane z=0 to the plane z=16 is:

```
> int(sqrt((2*t)^2+(6*t^2)^2+(6*t^2)^2), t=0..2); evalf(%);
```

$$\frac{73}{27}\sqrt{73} - \frac{1}{27} \qquad (32)$$

$$23.06341753$$

c) The trajectory crosses the plane x+y+z = 41 at t=2. The cosine of the angle at which it crosses (measured from the plane's normal) is the cosine of the angle between the velocity

```
> subs(t=2, vector([2*t, 6*t^2, 6*t^2]));
```

$$\left[\begin{array}{ccc} 4 & 24 & 24 \end{array}\right] \quad (33)$$

and the normal to the plane, [1,1,1]:

```
> dotprod([4,24,24], [1,1,1])/(sqrt(4^2+24^2+24^2)*sqrt(3)); evalf
  (%);
```

$$\frac{13}{219}\sqrt{73}\sqrt{3} \quad (34)$$

$$0.8784585920$$

The negative of this answer is also acceptable, if you chose the opposite orientation for the plane.

d) The curvature of the trajectory at t=2 is the magnitude of v X a / v^3, where the acceleration is

```
> subs(t=2, vector([2, 12*t, 12*t]));
```

$$\left[\begin{array}{ccc} 2 & 24 & 24 \end{array}\right] \quad (35)$$

```
> crossprod([4,24,24], [2,24,24])/(4^2 + 24^2 + 24^2)^(3/2);
```

$$\frac{1}{1364224}\left[\begin{array}{ccc} 0 & -48 & 48 \end{array}\right]\sqrt{1168} \quad (36)$$

Hence k =

```
> sqrt(2)*48*sqrt(1168)/1364224; evalf(%);
```

$$\frac{3}{21316}\sqrt{2}\sqrt{73} \quad (37)$$

$$0.001700560044$$

e) In the moving frame t,n,b, the acceleration is:

a(t) = s'' t + k s' n (no component with b)

```
> speed := t -> 2*sqrt(t^2+18*t^4);
```

$$speed := t \rightarrow 2\sqrt{t^2 + 18\,t^4} \quad (38)$$

```
> diff(speed(t), t);
```

$$\frac{2\,t + 72\,t^3}{\sqrt{t^2 + 18\,t^4}} \quad (39)$$

```
> subs(t=2, %); evalf(%);
```

$$\frac{145}{73}\sqrt{292} \quad (40)$$

$$33.94193269$$

This is the coefficient of the unit vector v. As for the component of n,

```
> (3/21316)*sqrt(2)*sqrt(73)*speed(2); evalf(%);
```

$$\frac{3}{73}\sqrt{2} \quad (41)$$

$$0.05811836556$$