Test 3 solutions (= version)

1. (15 points) Consider the vector field

```
> F:= (x,y) -> [1 + sin(y)^2, x*sin(2*y) + 1];
```

$$F := (x, y) \to \left[1 + \sin(y)^2, x \sin(2 y) + 1\right] \quad (1)$$

a) Is there a scalar function f(x,y) for which F = grad f ?
This requires that the partials

```
> diff(1 + sin(y)^2,y); diff(x*sin(2*y) + 1, x);
```

$$2 \sin(y) \cos(y) \quad (2)$$

$$\sin(2 y)$$

are equal, which they are, thanks to one of the trig identities you were reminded of.

b) If there is such a function, find it. If there is not, change the coefficient of i so that there is such a function:
There is. To find it we integrate P in x and Q in y. Maple does not add the +f(y) and + h(x) to these:

```
> int(1 + sin(y)^2,x); int(x*sin(2*y) + 1, y);
```

$$\left(1 + \sin(y)^2\right) x \quad (3)$$

$$-\frac{1}{2} x \cos(2 y) + y$$

Using the identity that cos(2 y) = 1 - 2 sin(y)^2, we need

x + x sin(y)^2 + f(y) = -x/2 + x sin(y)^2 + y + h(x)

By comparing the two sides we see that f(y) = y + constant and h = 3 x/2 + const. A suitable function f is therefore

```
> f := (x,y) -> x + y + x*sin(y)^2; diff(f(x,y),x); diff(f(x,y),y);
```

$$f := (x, y) \to x + y + x \sin(y)^2 \quad (4)$$

$$1 + \sin(y)^2$$

$$1 + 2 x \sin(y) \cos(y)$$

you could add any constant to this f and still be correct.

c) Write a differential equation the solutions of which are f(x,y) = c.
In differentials, the answer would be
(1 + sin(y)^2) dx + (x*sin(2*y) + 1) dy = 0
so in the form asked for we have
dy/dx = - (1 + sin(y)^2)/(x*sin(2*y) + 1) .

2. (15 points) Find the point on the surface z = sqrt(11 + x^2 + y^2) which is closest to (8,6,0). An equivalent constraint is g(x,y,z) = 11,
where

```
> g := (x,y,z) -> z^2 - x^2 - y^2;
```

$$g := (x, y, z) \rightarrow z^2 - x^2 - y^2 \quad (5)$$

a) A suitable objective function is the distance squared,

```
> f := (x,y,z) -> (x-8)^2 +(y-6)^2 + z^2;
```

$$f := (x, y, z) \rightarrow (x-8)^2 + (y-6)^2 + z^2 \quad (6)$$

b) The system of equations the solution of which answers this question is: grad f = lambda grad g and also: the constraint equation.

```
> with(linalg): grad(f(x,y,z), vector([x,y,z]));
```

$$[2x - 16 \quad 2y - 12 \quad 2z] \quad (7)$$

```
> grad(g(x,y,z), vector([x,y,z]));
```

$$[-2x \quad -2y \quad 2z] \quad (8)$$

so:

```
> solve({2*x-16 = -2*lambda*x,2*y-12 = -2*lambda*y, 2*z = 2*lambda*
  z, g(x,y,z) = 11}, {x,y,z, lambda});
```

$$\left\{z=0, y=\frac{3}{5}\,\mathrm{RootOf}\left(11+\_Z^2, label=\_L2\right), \lambda=-\frac{10}{11}\,\mathrm{RootOf}\left(11+\_Z^2, label=\_L2\right)-1, x=\frac{4}{5}\,\mathrm{RootOf}\left(11+\_Z^2, label=\_L2\right)\right\}, \{y=3, \lambda=1, z=6, x=4\}, \{y=3, \lambda=1, z=-6, x=4\} \quad (9)$$

The unit vector The possibilities with z=0 are not real numbers, so

c) The nearest point is (4,3,6) .

3. (15 points) Calculate the volume of the solid defined by:

$$0 \le z \le 9 - y \exp(2x) \quad , 1/2 \le x \le 1, -1 \le y \le 1$$

This is a double integral,

```
> int(9 - y*exp(2*x), x=1/2..1);
```

$$\frac{9}{2} - \frac{1}{2}\,y\,e^2 + \frac{1}{2}\,y\,e \quad (10)$$

```
> int(%, y=-1..1);
```

$$9 \quad (11)$$

4. (15 points - aren't you glad you did #16 on p. 980?) Let Ω be the triangle formed by the x-axis and the lines y= 3 x, and x = 2.

Calculate one of the following:

```
> Int(Int(exp(-x^2), x=y/3..2), y=0..6);
```

$$\int_{0}^{6}\int_{\frac{1}{3}y}^{2} e^{(-x^2)}\,dx\,dy \qquad (12)$$

or

```
> Int(Int(exp(-x^2), y=0..3*x), x=0..2);
```

$$\int_{0}^{2}\int_{0}^{3x} e^{(-x^2)}\,dy\,dx \qquad (13)$$

```
> int(int(exp(-x^2), y=0..3*x), x=0..2);
```

$$-\frac{3}{2}e^{(-4)}+\frac{3}{2} \qquad (14)$$