Test 3 solutions (= version)

1. (15 points) Find the point on the surface z = sqrt(3+ x^2 + y^2) which is closest to (6,4,0).
An equivalent constraint is g(x,y,z) = 3,
where

```
> g := (x,y,z) -> z^2 - x^2 - y^2;
```

$$g := (x, y, z) \rightarrow z^2 - x^2 - y^2 \quad (1)$$

a) A suitable objective function is the distance squared,

```
> f := (x,y,z) -> (x-6)^2 +(y-4)^2 + z^2;
```

$$f := (x, y, z) \rightarrow (x-6)^2 + (y-4)^2 + z^2 \quad (2)$$

b) The system of equations the solution of which answers this question is: grad f = lambda grad g
and also: the constraint equation.

```
> with(linalg): grad(f(x,y,z), vector([x,y,z]));
Warning, the protected names norm and trace have been redefined and unprotected
```

$$[2x - 12 \quad 2y - 8 \quad 2z] \quad (3)$$

```
> grad(g(x,y,z), vector([x,y,z]));
```

$$[-2x \quad -2y \quad 2z] \quad (4)$$

so:

```
> solve({2*x-12 = -2*lambda*x,2*y-8 = -2*lambda*y, 2*z = 2*lambda*z,
  g(x,y,z) = 3}, {x,y,z, lambda});
```

$$\left\{z=0, y=2\,\mathrm{RootOf}\left(3+13\,\_Z^2, label=\_L1\right), \lambda=-\frac{26}{3}\,\mathrm{RootOf}\left(3+13\,\_Z^2, label=\_L1\right)-1, x=3\,\mathrm{RootOf}\left(3+13\,\_Z^2, label=\_L1\right)\right\}, \{z=4, \lambda=1, x=3, y=2\}, \{z=-4, \lambda=1, x=3, y=2\} \quad (5)$$

The unit vector The possibilities with z=0 are not real numbers, so

c) The nearest point is (3,2,4) .

2. (15 points) Consider the vector field

```
> F:= (x,y) -> [y* sin(2*x) -1, 1 + sin(x)^2];
```

$$F := (x, y) \rightarrow \left[y \sin(2x) - 1, 1 + \sin(x)^2\right] \quad (6)$$

a) Is there a scalar function f(x,y) for which F = grad f ?
This requires that the partials

```
> diff(y* sin(2*x) -1,y); diff(1 + sin(x)^2, x);
```

$$\sin(2x) \quad (7)$$

$$2 \sin(x) \cos(x)$$

are equal, which they are, thanks to one of the trig identities you were reminded of.

b) If there is such a function, find it. If there is not, change the coefficient of i so that there is such a function:

There is. To find it we integrate P in x and Q in y. Maple does not add the +f(y) and + h(x) to these:

```
> int(y* sin(2*x) -1,x); int(1 + sin(x)^2, y);
```

$$-\frac{1}{2}y\cos(2x)-x \quad (8)$$

$$\left(1+\sin(x)^2\right)y$$

Using the identity that cos(2 y) = 1 - 2 sin(y)^2, we need

-x -y/2 + y sin(x)^2 + f(y) = y + y sin(x)^2 + h(x)

By comparing the two sides we see that f(y) = 3y/2 + constant and h = -x + const. A suitable function f is therefore

```
> f := (x,y) -> -x + y + y*sin(x)^2; diff(f(x,y),x); diff(f(x,y),y);
```

$$f := (x,y)\rightarrow -x+y+y\sin(x)^2 \quad (9)$$

$$-1+2y\sin(x)\cos(x)$$

$$1+\sin(x)^2$$

3. (15 points) Calculate the volume of the solid defined by:

$$0 \le z \le 3+\sin(\text{Pi } y) \quad , -1 \le x \le 2,\ 1 \le y \le 3$$

This is a double integral,

```
> int(3 + sin(Pi*y), x=-1..2);
```

$$9+3\sin(\pi y) \quad (10)$$

```
> int(%, y=1..3);
```

$$18 \quad (11)$$

4. (15 points - aren't you glad you did #16 on p. 980?) Let Ω be the triangle formed by the y-axis and the lines y= 2 x, and y = 3.

Calculate one of the following:

```
> Int(Int(exp(-y^2), x=0..y/2), y=0..3);
```

$$\int_0^3 \int_0^{\frac{1}{2}y} e^{\left(-y^2\right)}\,dx\,dy \quad (12)$$

or

```
> Int(Int(exp(-y^2), y=2*x..3), x=0..3/2);
```

$$\int_{0}^{\frac{3}{2}} \int_{2x}^{3} e^{\left(-y^2\right)} \, dy \, dx \qquad (13)$$

```
> int(int(exp(-y^2), y=2*x..3), x=0..3/2);
```

$$\frac{1}{4} - \frac{1}{4} e^{(-9)} \qquad (14)$$

```
>
```