Test 4 solutions (beige version)

1. (15 points) Find the volume of the tetrahedron with

x + y/4 - 2 z > 1
0 < x < 1
0 < y < 4,
0 < z

Note: If it is projected into the x-y plane, it becomes a triangle in the first quadrant, bounded by x=1, y=4, and y=-4x + 4. We can write the volume as a two variable integral over this triangle (for example). The height at point (x,y) will be x/2 + y/8 - 1/2

An explicit integral for the answer is:

```
> Int(Int(x/2 + y/8 - 1/2, y=4-4*x..4), x=0..1);
```

$$\int_0^1 \int_{4-4x}^4 \frac{1}{2}x + \frac{1}{8}y - \frac{1}{2}\, dy\, dx \qquad (1)$$

The value of the integral is:

```
> int(x/2 + y/8 - 1/2, y=4-4*x..4);
```

$$2x^2 + 1 - \frac{1}{16}(4-4x)^2 - 2x \qquad (2)$$

```
> int(%, x=0..1);
```

$$\frac{1}{3} \qquad (3)$$

2. (15 points) Determine whether

h := (3 x + y^2)i + (3 y - x^2) j

is a gradient, then calculate the line integral of h over the curve,

C := the circular arc x^2 + y^2 = 16, y ≥ 0 from (4,0) to (0,4)

a and b) Is h a gradient? Le's do the calculation:

```
> P := (x,y) -> 3 * x + y^2 : Q := (x,y) -> 3*y - x^2 :
> diff(P(x,y),y) - diff(Q(x,y),x);
```

$$2y + 2x \qquad (4)$$

This is not zero. The answer is NO. Therefore we will need to parametrize the curve, which can be done in many ways, for example, x = 4 cos(t), y = 4 sin(t), t from 0 to pi/2

c) Evaluate the line integral:

```
> int((3*4*cos(t)+(4*sin(t))^2)*(-4*sin(t)) + (3*4*sin(t) - (4*cos
  (t))^2)*(4*cos(t)), t=0..Pi/2);
```

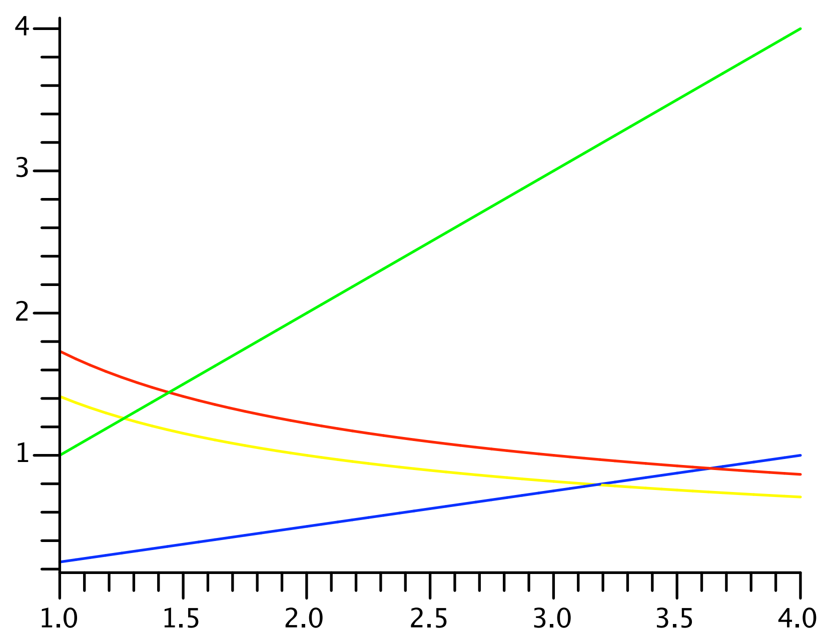
$$-\frac{256}{3} \quad (5)$$

3. (15 points) Consider the region such that

x > 0, y > 0
2 < x y^2 < 3
x/4 < y < x

a) Sketch the region here:

```
> plot({sqrt(2/x),sqrt(3/x),x/4,x}, x=1..4);
```



b) Find a variable transformation to "straighten" the bounding curves:

```
> u := (x,y) -> x*y^2; v := (x,y) -> y/x;
```

$$u := (x, y) \rightarrow x\, y^2 \quad (6)$$

$$v := (x, y) \rightarrow \frac{y}{x}$$

c) Express x and y in terms of the new variables;

The inverse functions are

```
> x := (u,v) -> u^(1/3)*v^(-2/3); y := (u,v) -> (u*v)^(1/3);
```

(7)

$$x := (u, v) \rightarrow \frac{u^{(1/3)}}{v^{(2/3)}} \quad (7)$$

$$y := (u, v) \rightarrow (u\, v)^{(1/3)}$$

d=e) Express the area as an explicit integral in u,v

```
> with(linalg):
```

Warning, the protected names norm and trace have been redefined and unprotected (8)

[*BlockDiagonal*, *GramSchmidt*, *JordanBlock*, *LUdecomp*, *QRdecomp*, *Wronskian*, *addcol*, *addrow*, *adj*, *adjoint*, *angle*, *augment*, *backsub*, *band*, *basis*, *bezout*, *blockmatrix*, *charmat*, *charpoly*, *cholesky*, *col*, *coldim*, *colspace*, *colspan*, *companion*, *concat*, *cond*, *copyinto*, *crossprod*, *curl*, *definite*, *delcols*, *delrows*, *det*, *diag*, *diverge*, *dotprod*, *eigenvals*, *eigenvalues*, *eigenvectors*, *eigenvects*, *entermatrix*, *equal*, *exponential*, *extend*, *ffgausselim*, *fibonacci*, *forwardsub*, *frobenius*, *gausselim*, *gaussjord*, *geneqns*, *genmatrix*, *grad*, *hadamard*, *hermite*, *hessian*, *hilbert*, *htranspose*, *ihermite*, *indexfunc*, *innerprod*, *intbasis*, *inverse*, *ismith*, *issimilar*, *iszero*, *jacobian*, *jordan*, *kernel*, *laplacian*, *leastsqrs*, *linsolve*, *matadd*, *matrix*, *minor*, *minpoly*, *mulcol*, *mulrow*, *multiply*, *norm*, *normalize*, *nullspace*, *orthog*, *permanent*, *pivot*, *potential*, *randmatrix*, *randvector*, *rank*, *ratform*, *row*, *rowdim*, *rowspace*, *rowspan*, *rref*, *scalarmul*, *singularvals*, *smith*, *stackmatrix*, *submatrix*, *subvector*, *sumbasis*, *swapcol*, *swaprow*, *sylvester*, *toeplitz*, *trace*, *transpose*, *vandermonde*, *vecpotent*, *vectdim*, *vector*, *wronskian*]

```
> det([[diff(x(u,v),u),diff(x(u,v),v)],[diff(y(u,v),u),diff(y(u,v),
  v)]]);
```

$$\frac{1}{3} \frac{u^{(1/3)}}{v^{(2/3)} (u\, v)^{(2/3)}} \quad (9)$$

```
> J := (u,v) -> (1/3)*u^(-1/3) *v^(-4/3);
```

$$J := (u, v) \rightarrow \frac{1}{3} \frac{1}{u^{(1/3)} v^{(4/3)}} \quad (10)$$

The area is

```
> Int(Int(J(u,v), u=2..3), v=1/4..1);
```

$$\int_{\frac{1}{4}}^{1} \int_{2}^{3} \frac{1}{3} \frac{1}{u^{(1/3)} v^{(4/3)}} \, du \, dv \quad (11)$$

```
> int(J(u,v), u=2..3);
```

$$\frac{1}{3} \frac{2^{(2/3)} \left(\frac{3}{4} 3^{(2/3)} 2^{(1/3)} - \frac{3}{2} \right)}{v^{(4/3)}} \quad (12)$$

```
> int(%, v=1/4..1);
```

$$\frac{1}{8}\,3^{(2/3)}\left(12\,2^{(2/3)}-12\right)-\frac{1}{8}\,2^{(2/3)}\left(12\,2^{(2/3)}-12\right) \tag{13}$$

```
> simplify(%); evalf(%);
```

$$\frac{3}{2}\,3^{(2/3)}\,2^{(2/3)}-\frac{3}{2}\,3^{(2/3)}-3\,2^{(1/3)}+\frac{3}{2}\,2^{(2/3)} \tag{14}$$

$$0.434103568$$

4. Evaluate the integral of z over the region Omega that lies

i) above the x-y plane
ii) above the cone z^2 = x^2 + y^2
iii) outside the vertical circular cylinder (like a tin-can sitting upright on the table) of radius 2, centered at the origin
iv) inside the sphere of radius 4, centered at the origin

a) Set the integral up explicitly in any of the 3-dimensional coordinate systems we have used in class. (You choose which one):

Cartesian is the worst choice, because the x-y integration region is a circular disk with another circular disk removed. You could write the intergral as a difference:

```
> Int(Int(Int(z, z=sqrt(x^2+y^2)..sqrt(16-x^2-y^2)),x=-sqrt(8-y^2)..
  sqrt(8-y^2)), y=-sqrt(8)..sqrt(8)) - Int(Int(Int(z, z=sqrt(x^2+
  y^2)..sqrt(16-x^2-y^2)),x=-sqrt(4-y^2)..sqrt(4-y^2)), y=-sqrt(4)..
  sqrt(4));
>
```

$$\int_{-2\sqrt{2}}^{2\sqrt{2}}\int_{-\sqrt{8-y^2}}^{\sqrt{8-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} z\,dz\,dx\,dy-\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} z\,dz\,dx\,dy \tag{15}$$

In cylindrical it is:

```
> Int(Int(Int(z*r, z=r..sqrt(16-r^2)), r=2..sqrt(8)), theta=0..2*Pi)
  ;
```

$$\int_{0}^{2\pi}\int_{2}^{2\sqrt{2}}\int_{r}^{\sqrt{16-r^2}} z\,r\,dz\,dr\,d\theta \tag{16}$$

To evaluate this one:

```
> 2*Pi*int(z*r, z=r..sqrt(16-r^2));
```

$$\pi\,r\left(16-2\,r^2\right) \tag{17}$$

```
> int(%, r=2..sqrt(8));
```

$$8\,\pi \tag{18}$$

In spherical it is:

```
> Int(Int(Int((rho*cos(phi))*rho^2*sin(phi), rho=2/sin(phi)..4),
  phi=Pi/6..Pi/4), theta=0..2*Pi);
```

(19)

$$\int_{0}^{2\pi}\int_{\frac{1}{6}\pi}^{\frac{1}{4}\pi}\int_{\frac{2\cdot 1}{\sin(\phi)}}^{4} \rho^3 \cos(\phi)\sin(\phi)\, d\rho\, d\phi\, d\theta \qquad (19)$$

The evaluation had better be the same:

```
> 2*Pi*int((rho*cos(phi))*rho^2*sin(phi), rho=2/sin(phi)..4);
```

$$\frac{1}{2}\pi\cos(\phi)\sin(\phi)\left(256-\frac{16}{\sin(\phi)^4}\right) \qquad (20)$$

```
> int(%, phi=Pi/6..Pi/4);
```

$$8\pi \qquad (21)$$

```
>
```