MATH 2401 - Harrell

Just what *is* curvature, anyway?

Lecture 5

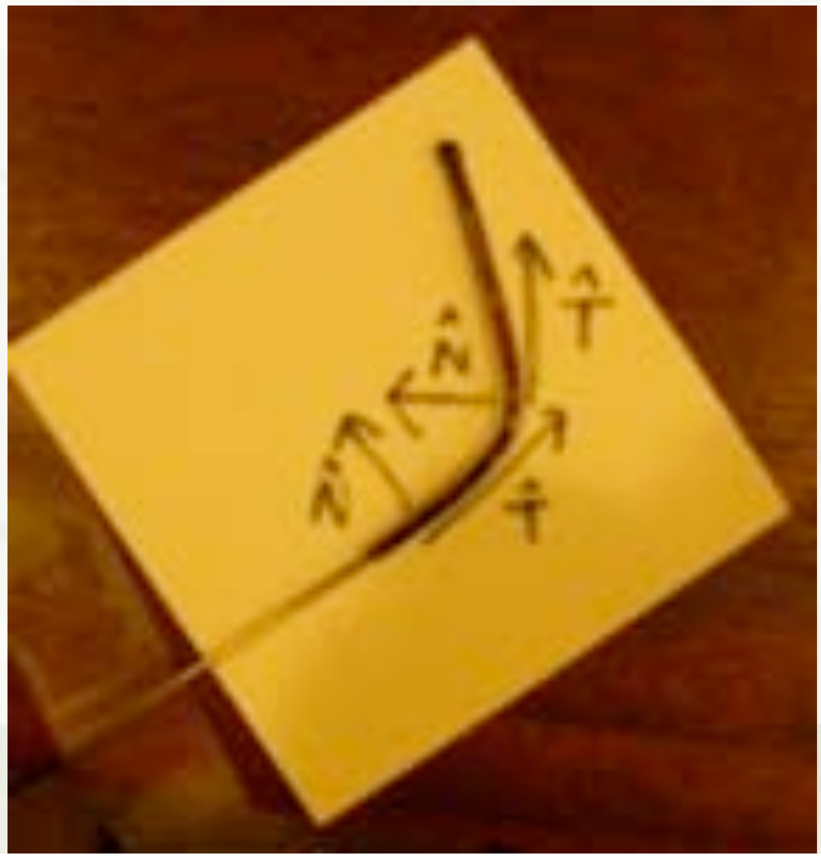
The osculating plane

- Bits of curve have a “best plane.”

stickies on wire.

Each stickie contains **T** *and* **N**.

Close-up



The osculating plane

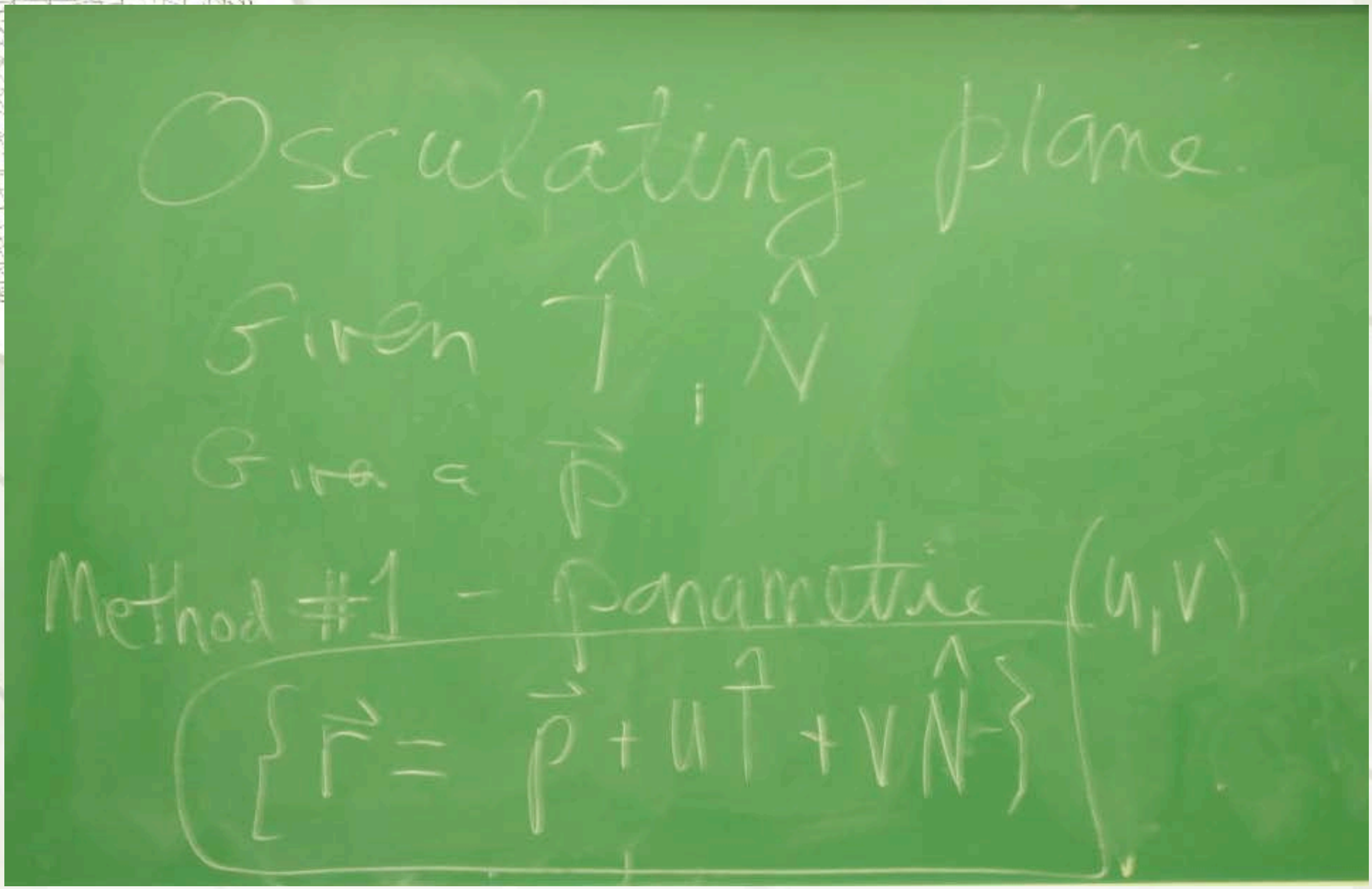
- Bits of curve have a “best plane.”
- One exception - a straight line lies in infinitely many planes.

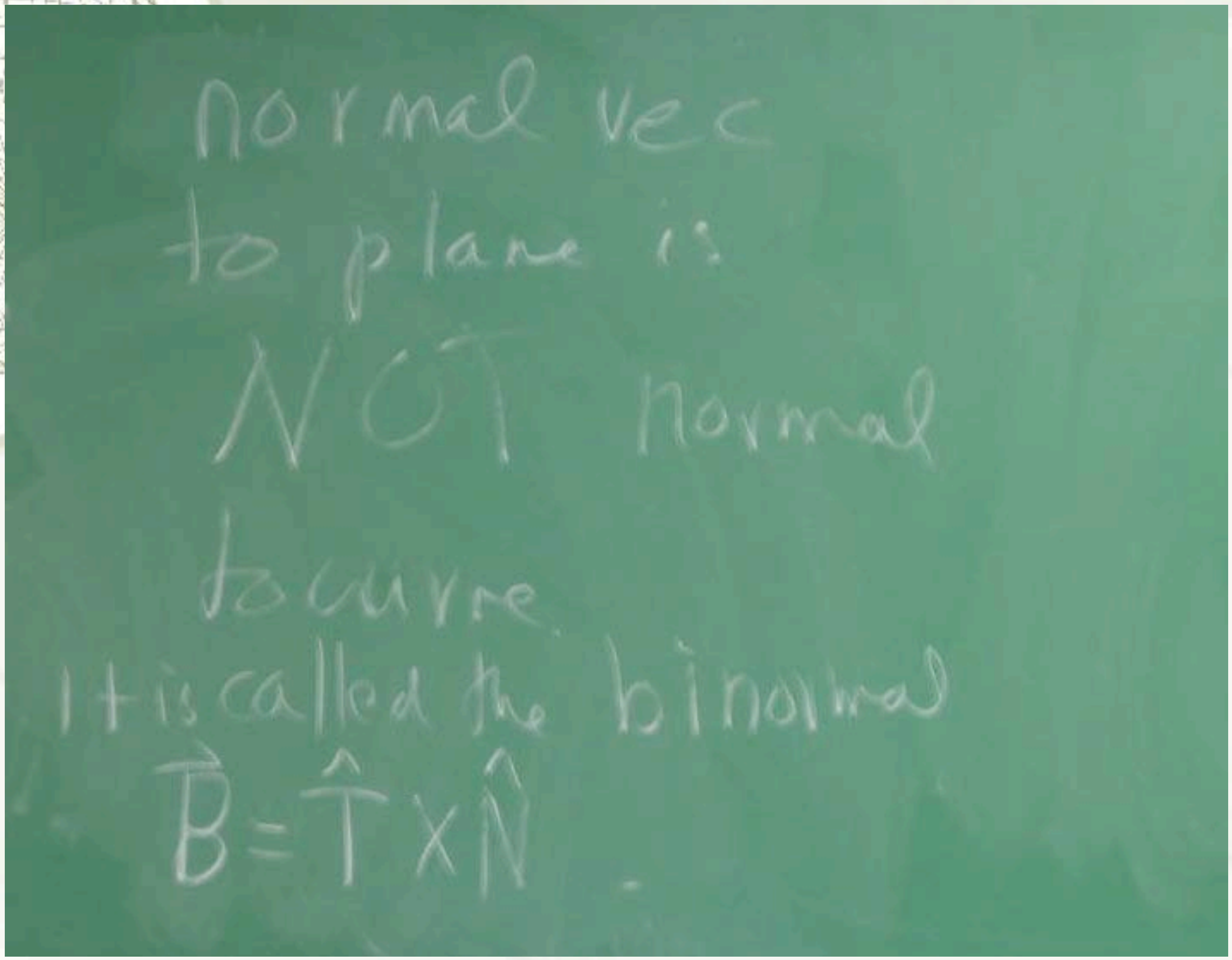
The osculating plane

- What's the formula, for example for the helix?

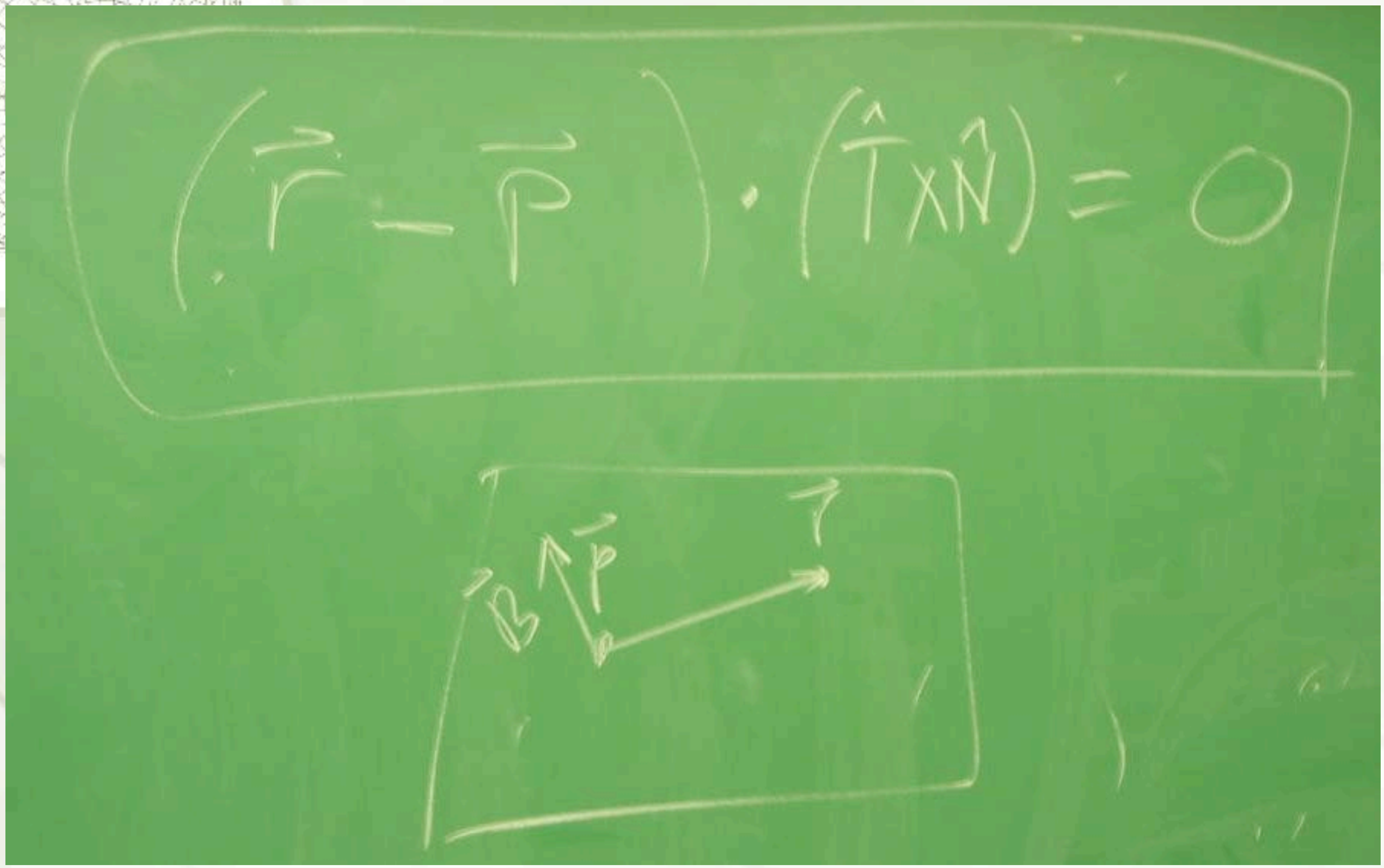
1. Parametric form

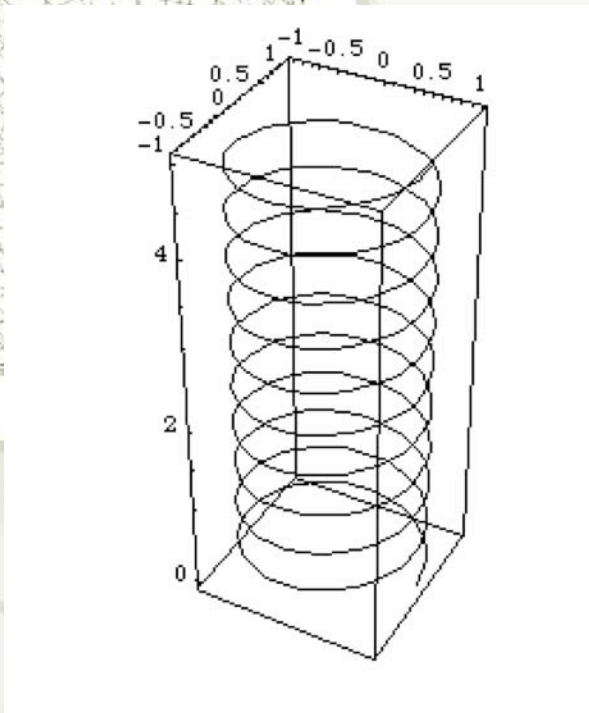
2. Single equation


Osculating plane
Given $\hat{T}$, $\hat{N}$
Given a $\vec{p}$
Method #1 - parametric (u,v)
$\{\vec{r} = \vec{p} + u\hat{T} + v\hat{N}\}$



N is normal (⊥) to the curve. It's **B** that's normal to the osculating plane. And of course **B** ⊥ **N**, because of the ×.


$(\vec{r} - \vec{p}) \cdot (\hat{T} \times \hat{N}) = 0$
$\vec{B}$
$\vec{p}$
$\vec{r}$



Example: The helix

$\mathbf{r}(t) = \cos(4\pi t)\,\mathbf{i} + \sin(4\pi t)\,\mathbf{j} + t\,\mathbf{k}$

$\mathbf{T}(t) = (-4\pi \sin(4\pi t)\,\mathbf{i} + 4\pi \cos(4\pi t)\,\mathbf{j} + \mathbf{k})/(1+16\pi^2)^{1/2}$

$\mathbf{N}(t) = -\cos(4\pi t)\,\mathbf{i} - \sin(4\pi t)\,\mathbf{j}$

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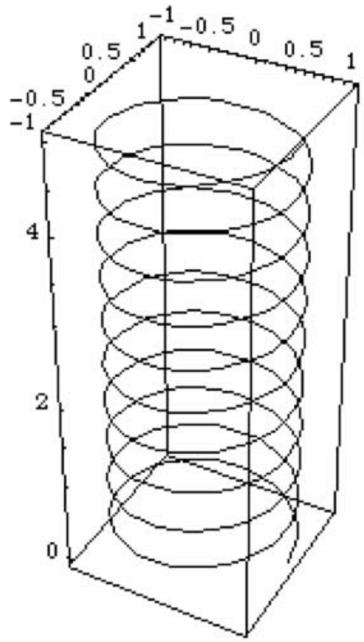


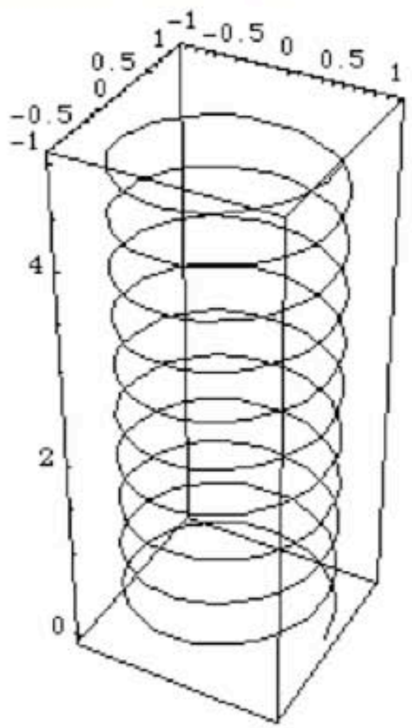
$\mathbf{r}(t) = \cos(4\pi t)\,\mathbf{i} + \sin(4\pi t)\,\mathbf{j} + t\,\mathbf{k}$

$\mathbf{T}(t) \times \mathbf{N}(t) = (-\sin(4\pi t)\,\mathbf{i} + \cos(4\pi t)\,\mathbf{j} - 4\pi\,\mathbf{k})/(1+16\pi^2)^{1/2}$

Osculating plane at (1,0,1): Calculate at t=1.

$(\mathbf{r}_{osc} - (\mathbf{i} + \mathbf{k})) \cdot (1\,\mathbf{j} - 4\pi\,\mathbf{k}) = 0$

(The factor $(1+16\pi^2)^{1/2}$ can be dropped.)



Example: The helix

In coordinates,

$$x_{helix}(t) = \cos(4\,\pi\,t),$$

$$y_{helix}(t) = \sin(4\,\pi\,t)$$

$$z_{helix}(t) = t$$

And

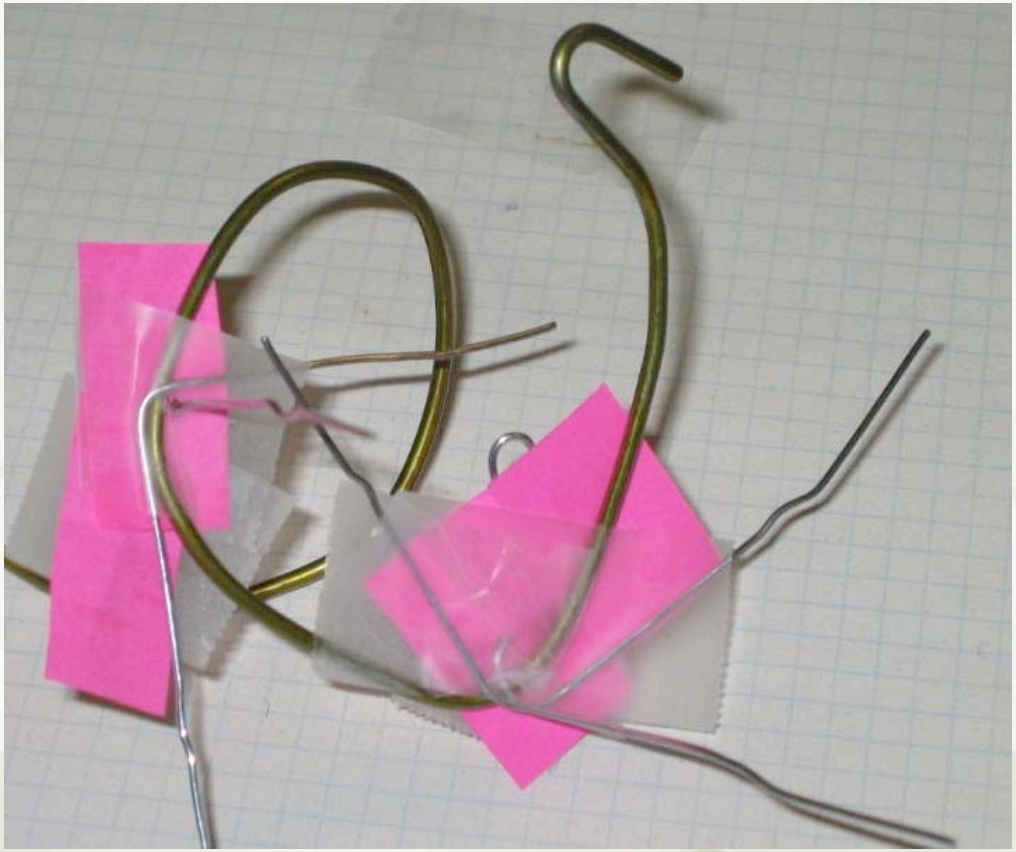
$$(x_{osc} - 1)\cdot 0 + (y_{osc} - 0)\cdot 1 + (z_{osc} - 1)\cdot(-4\pi) = 0,$$

Which simplifies to:

$$\mathbf{y_{osc} - 4\,\pi\,z_{osc} = -\,4\,\pi}$$

The moving trihedron

- The curve's preferred coordinate system is oriented along (**T**,**N**,**B**), *not* some Cartesian system (**i**,**j**,**k**) in the sky.



The moving trihedron

- An vehicle can rotate around any of these axes. A rotation around **T** is known as *roll*. If the vehicle has wings (or a hull) it may prefer a second direction over **N**. For example, the wing direction may correlate with **N** when the airplane turns without raising or lowering the nose. Such an acceleration is called *yaw*.

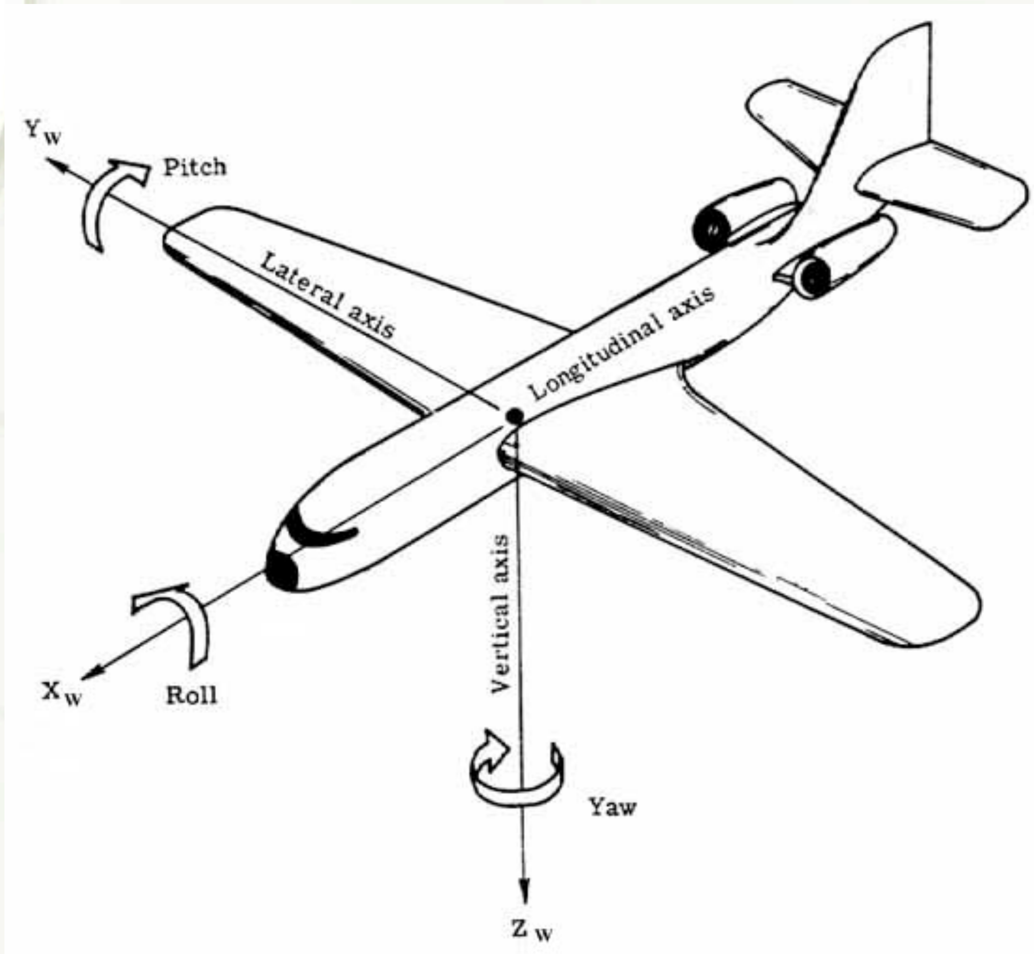


Figure from JPL/NASA

The moving trihedron

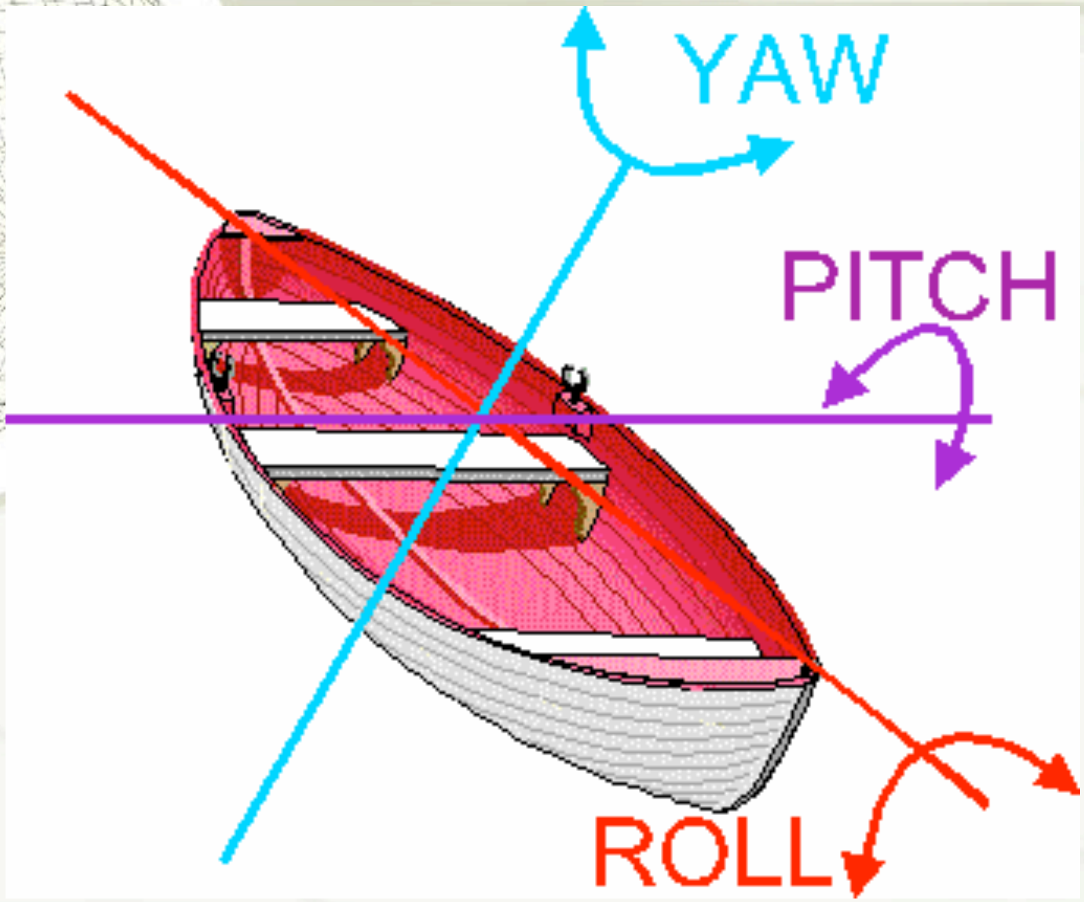
- However, when the aircraft soars or dives (this kind of acceleration is called *pitch*), the normal vector **N** is perpendicular to the wing axis, which in this case correlates with the binormal **B**.

- An aircraft can accelerate, roll, yaw, and pitch all at once.

The moving trihedron

- However, when the aircraft soars or dives (this kind of acceleration is called *pitch*), the normal vector **N** is perpendicular to the wing axis, which in this case correlates with the binormal **B**.

- An aircraft can accelerate, roll, yaw, and pitch all at once. *Fasten your seatbelt!*



Watercraft have the same kinds of accelerations as aircraft. The rudder controls yaw. The boat is usually designed to minimize pitch and roll.

Figure from boatsafe.com .

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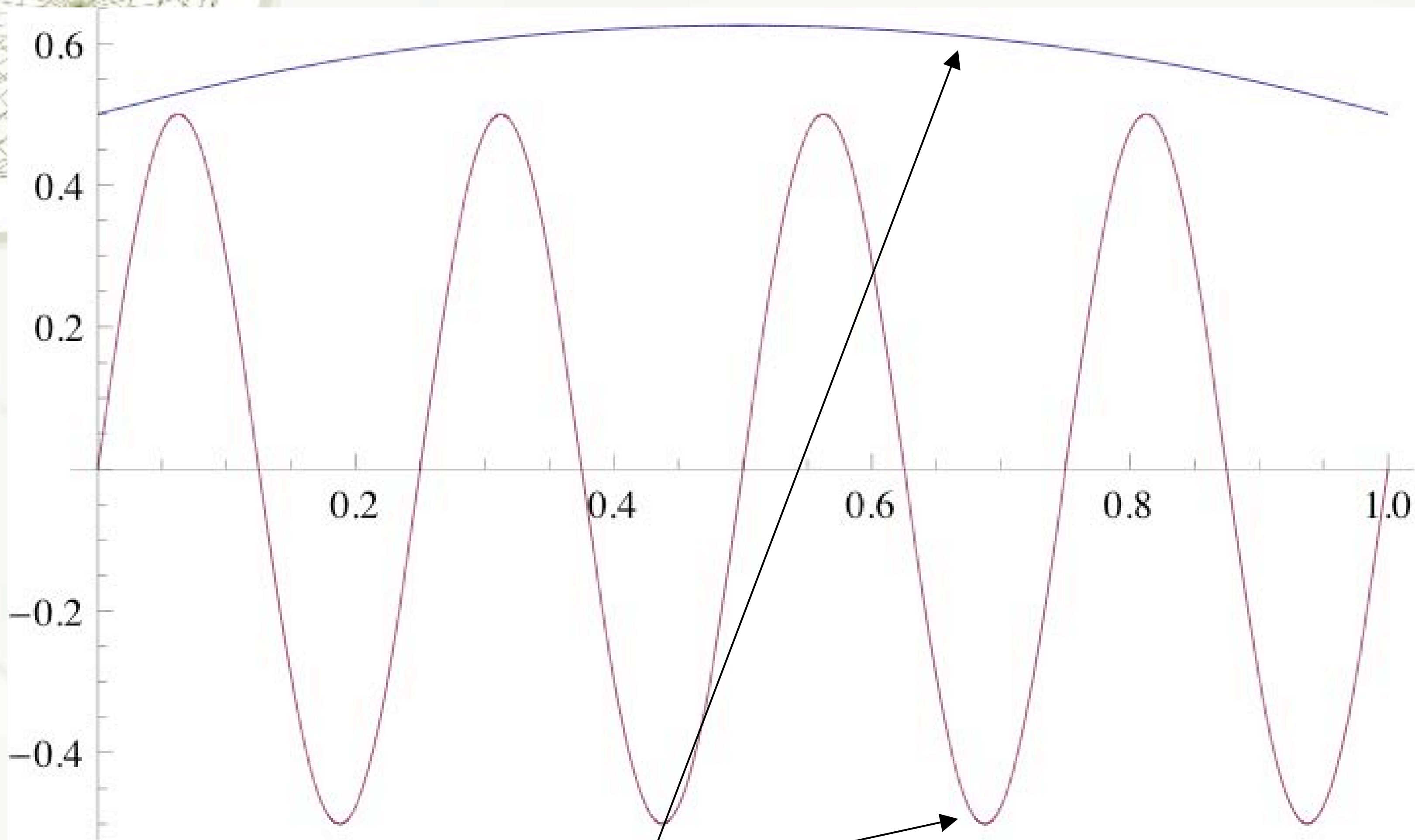
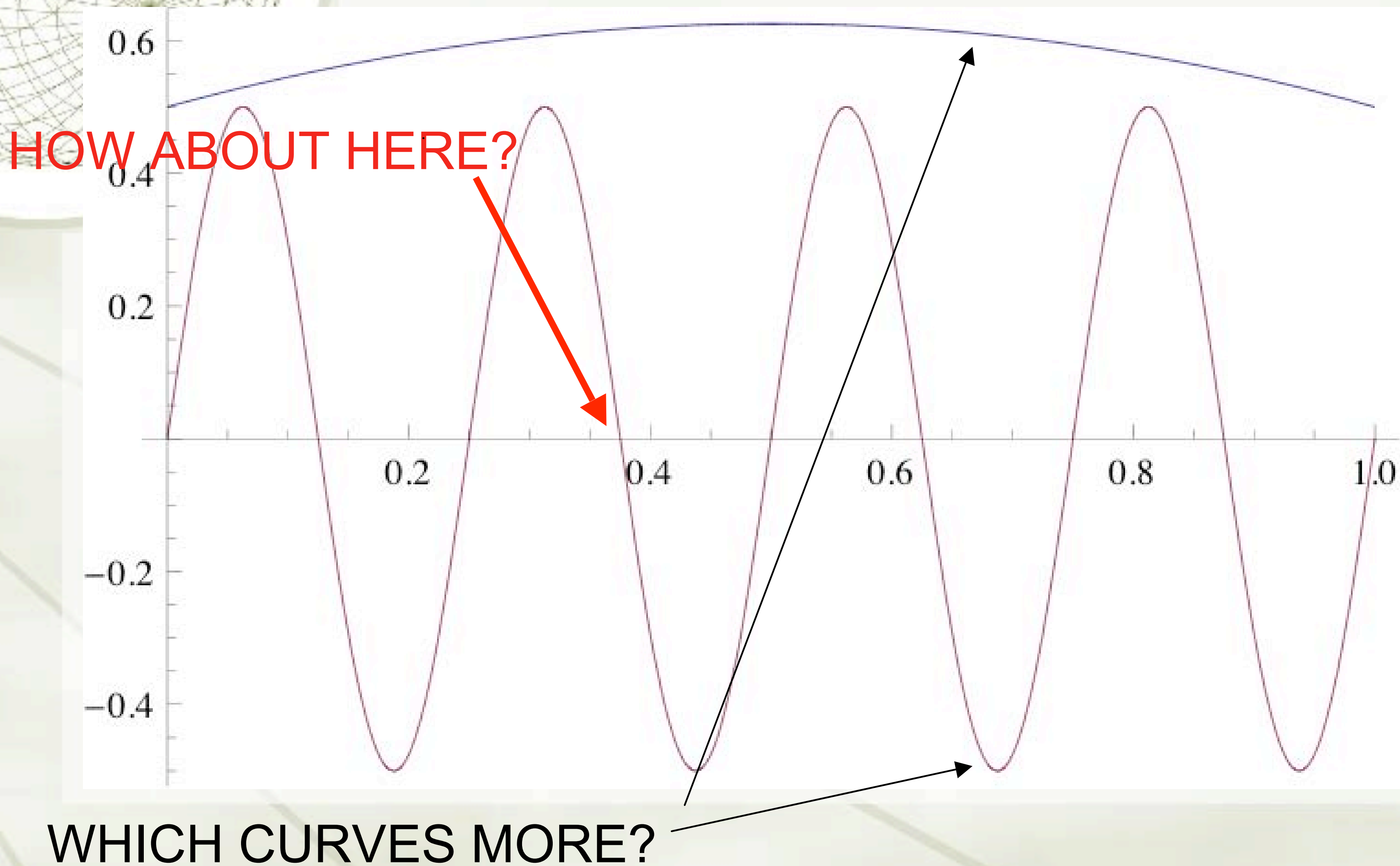
- How do you know a curve is curving? And how much?
- The answer should depend just on the shape of the curve, not on the speed at which it is drawn. *So it connects with arclength s, not with a time-parameter t.*

Just what is *curvature?*

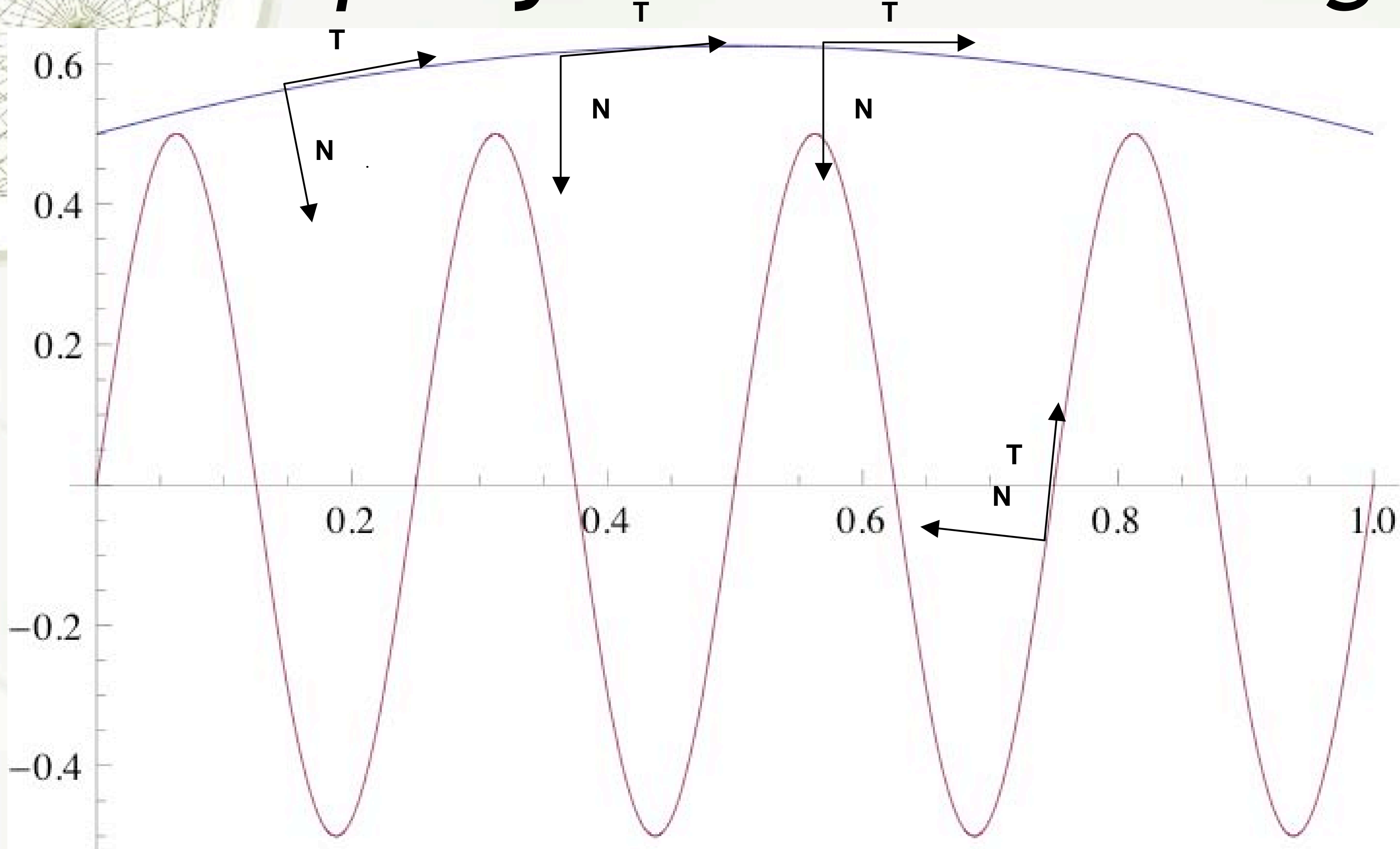


WHICH CURVES MORE?

Just what is *curvature?*



How rapidly do T and N change?



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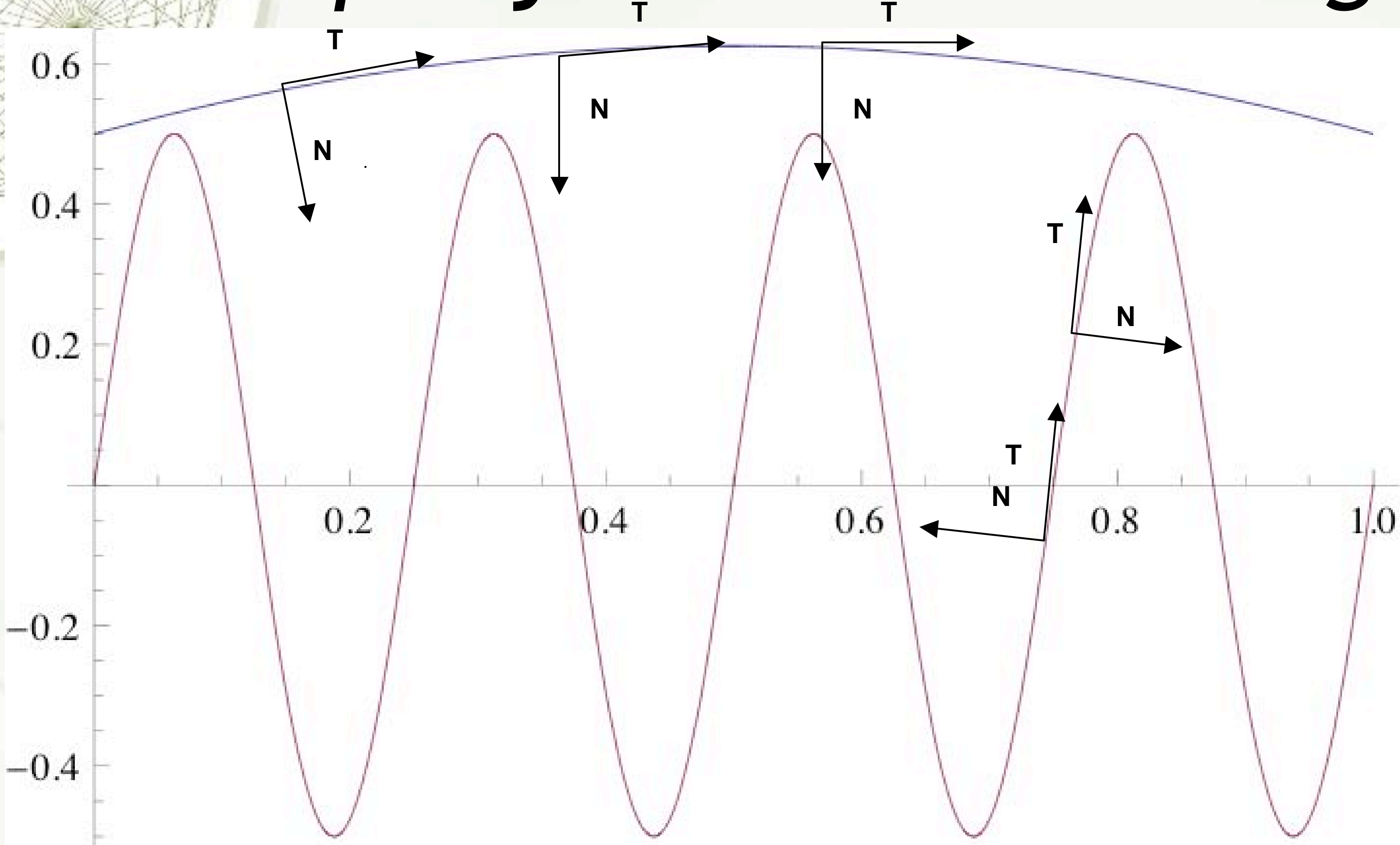

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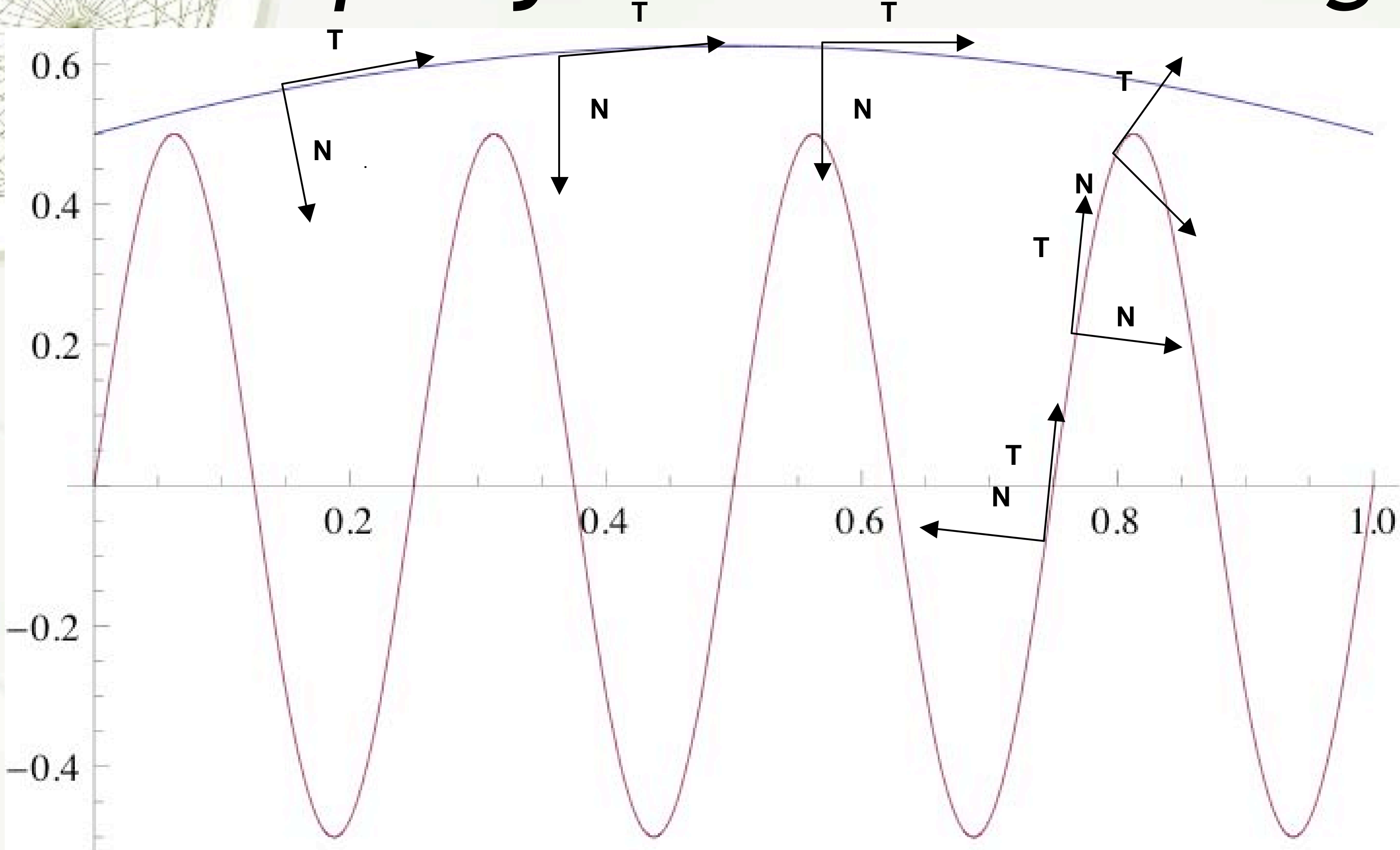

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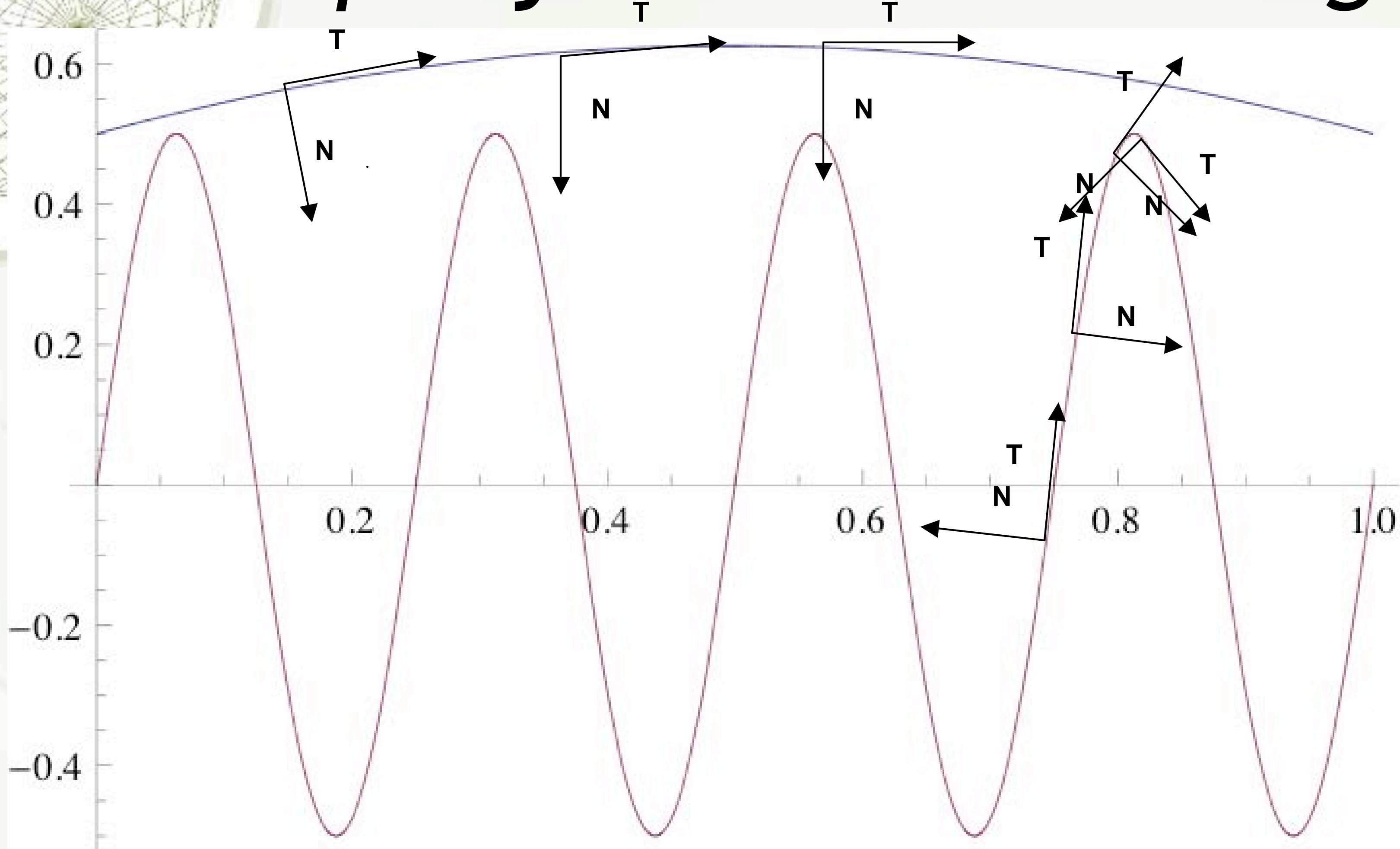

Just what *is* curvature?

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  - 2D: How about $|d\phi/ds|$, where ϕ is the direction of **T** with respect to the x-axis?

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- And let's be quantitative about it!
  - 2D: How about $|d\phi/ds|$, where ϕ is the direction of **T** with respect to the x-axis?
  - Conceptually that's reasonable. In practice it is just a bit involved…

Just what *is* curvature?

- And let's be quantitative about it!
  - 2D: How about $|d\phi/ds|$, where ϕ is the direction of **T** with respect to the x-axis?
  - To get started, notice that the direction of **T** is the same as that of the tangent line. That is,

$$\tan \phi = dy/dx = (dy/ds)/(dx/ds)$$

(fasten seatbelts for the next slide!)

A tricky calculation of $K = \frac{d\phi}{ds}$

$$\tan\phi = \frac{dy}{dx} = (dy/ds)\big/(dx/ds)$$

$$\boxed{\frac{d}{ds}\tan\phi = \frac{y''x' - y'x''}{(x')^2}}$$ by quotient rule

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$$= \sec^2\phi\,\frac{d\phi}{ds} = (1+\tan^2\phi)\frac{d\phi}{ds}$$

$$= \left(1 + \left(y'/x'\right)^2\right)\underbrace{d\phi/ds}_{K} = \frac{(x')^2+(y')^2}{(x')^2}$$

(at least up to ±)

It’s our old friend the chain rule, used in a creative way!

A tricky calculation of $K = \frac{d\phi}{ds}$

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K (at least up to $\pm$)

Solving:

$$K = \left| \frac{y''x' - y'x''}{(x')^2+(y')^2} \right|$$

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K (at least up to $\pm$)

Solving:

$$K = \frac{y''x' - y'x''}{\underset{=1}{\cancel{(x')^2+(y')^2}}}$$

Different expressions for κ

- $\kappa = |d\phi/ds|$

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- $\kappa = |x'(s)\, y''(s) - y'(s)\, x''(s)|$
- $\kappa = \dfrac{|x'(t)\, y''(t) - y'(t)\, x''(t)|}{|(x'(t))^2 + (y'(t))^2|^{3/2}}$

Huh??

Example

- Circle of radius 5.

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  - *No calculus needed!*

Example

- Circle of radius 5.
  - *No calculus needed!*
  - *If you move distance Δs along the perimeter, the change in angle is $\Delta s/5$. So $\Delta\phi/\Delta s = 1/5$.* The general rule for a circle is that the curvature is the reciprocal of the radius.

Example

- Spiral: The formula for curvature is complicated, but the spiral is simple, so the curvature should be simple.

Example

- Spiral: The formula for curvature is complicated, but the spiral is simple, so the curvature should be simple.
- Still, we’ll be lazy and use Mathematica:

Example

```
In[5]:= Speed[r_] := Sqrt[D[r[[1]], t]^2 + D[r[[2]], t]^2]

In[22]:= NumeratorOfCurvature[r_] :=
          D[r[[2]], {t, 2}] D[r[[1]], t] - D[r[[1]], {t, 2}] D[r[[2]], t]

         Curvature[r_] := D[r[[2]], {t, 2}] D[r[[1]], t] -
           D[r[[1]], {t, 2}] D[r[[2]], t] / Speed[r]^3

In[11]:= Spiral := {t Cos[t], t Sin[t]}

In[12]:= Speed[Spiral]
```

Out[12]= $\sqrt{(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])^2+(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])^2}$

```
In[19]:= Curvature[Spiral]
```

Out[19]= $(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])\,(2\,\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])-\dfrac{(-t\,\mathrm{Cos}[t]-2\,\mathrm{Sin}[t])\,(t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])}{\left((t\,\mathrm{Cos}[t]+\mathrm{Sin}[t])^2+(\mathrm{Cos}[t]-t\,\mathrm{Sin}[t])^2\right)^{3/2}}$

```
In[23]:= Simplify[NumeratorOfCurvature[Spiral]]
```

Out[23]= $2+t^2$

```
In[24]:= Simplify[Speed[Spiral]]
```

Out[24]= $\sqrt{1+t^2}$

```
In[25]:= %% / %^3
```

Out[25]= $\dfrac{2+t^2}{\left(1+t^2\right)^{3/2}}$

Dimensional analysis

- What units do you use to measure curvature?

Dimensional analysis

- What units do you use to measure curvature?
  - Hint: angles are considered dimensionless, since radian measure is a ratio of arclength (cm) to radius (also cm)

Dimensional analysis

- What units do you use to measure curvature?
  - Answer: 1/distance, for instance 1/cm.

$1/\kappa$ is known as the *radius of curvature*. It's the radius of the circle that best matches the curve at a given contact point.